CONFORMAL MAPPING OF SURFACES

By H. Randolph Pyle

1. **Introduction.** The object of this paper is to discuss the conditions which must be satisfied by the equations which map any surface conformally on any other surface. The classical solution of the problem expresses these conditions in terms of equations of the second degree. It is our purpose to show that these may be reduced to linear relationships. From these equations an analogue of Laplace's equation is obtained.

2. **Metrics.** The usual form for the metric of a surface is

$$ds^2 = E\,dx^2 + 2F\,dx\,dy + G\,dy^2,$$

where E, F, and G are functions of x and y. If we set

$$a = E/H, \qquad c = G/H, \qquad b = F/H,$$

where $H^2 = EG - F^2$, we have $ac - b^2 = 1$, so that c is determined when a and b are given.

Any surface S has the metric

$$ds^2 = H(a\,dx^2 + 2b\,dx\,dy + c\,dy^2), \qquad ac - b^2 = 1. \tag{2.1}$$

3. **Conditions for conformal mapping.** If we are given the surface S with metric (2.1) and the surface S' with metric

$$ds'^2 = H'(a'\,dx'^2 + 2b'\,dx'\,dy' + c'\,dy'^2), \qquad a'c' - b'^2 = 1 \tag{3.1}$$

and the point (x', y') of S' is related to the point (x, y) of S by the transformation $x' = \varphi(x, y)$, $y' = \psi(x, y)$, we have $d\varphi = \varphi_x\,dx + \varphi_y\,dy$, $d\psi = \psi_x\,dx + \psi_y\,dy$, $a' = a'(\varphi, \psi)$, $b' = b'(\varphi, \psi)$. Here we make the usual assumptions concerning continuity and the existence of the partial derivatives of φ and ψ. The necessary and sufficient condition for the conformal mapping of S on S' is $ds'^2 = \lambda^2\,ds^2$.

It will be convenient to use the well-known differential parameters of the first order:

$$\Delta_1\varphi = [a\varphi_y^2 - 2b\varphi_x\varphi_y + c\varphi_x^2]/H, \qquad \Delta_1\psi = [a\psi_y^2 - 2b\psi_x\psi_y + c\psi_x^2]/H,$$

$$\Delta_1(\varphi, \psi) = [a\varphi_y\psi_y - b(\varphi_x\psi_y + \varphi_y\psi_x) + c\varphi_x\psi_x]/H, \tag{3.2}$$

$$\theta(\varphi, \psi) = J/H,$$

where $J = \varphi_x\psi_y - \varphi_y\psi_x$.

If the surface S is referred to the parameters x', y' and the condition for con-

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