A CHARACTERISTIC CONDITION FOR SEMI-PRIMARY RINGS

By Jakob Levitzki

In the present note we shall use the term *radical* for the sum R of all two-sided nilpotent ideals of a ring S. If R is nilpotent and the quotient-ring S/R is semi-simple, the ring S is called semi-primary. It has been proved [1] that if S is a ring with minimal condition for left ideals, then S is semi-primary. But neither this condition nor the weaker assumptions which were found later by other authors are necessarily satisfied by each semi-primary ring. In the following theorem, necessary and sufficient minimal conditions for semi-primary rings are stated.

Theorem. *A ring S is semi-primary if and only if the following conditions are satisfied:*

(1) *Each descending chain $L_1 \supset L_2 \supset \cdots$, where the L_i are left ideals of S containing the radical R, is finite.*

(2) *Each descending chain of the form: $A_1 \supset A_1 \cdot A_2 \supset A_1 \cdot A_2 \cdot A_3 \supset \cdots$, where the A_i are two-sided ideals contained in the radical R, is finite.*

Proof. If S is semi-primary, then, as is well known, (1) is satisfied, and as is easily seen, (2) is valid also. In fact, supposing that $R^n = 0$, we have for $m \geq n$ also $A_1 \cdot A_2 \cdots A_m = 0$, i.e., (2) is valid. To prove the second part of the theorem, we only have to show that from (2) follows the nilpotency of R, since, as is well known, S/R is then semi-simple by (1). To prove that R is nilpotent, first note that from $R \supseteq R^2 \supseteq R^3 \supseteq \cdots$ follows by (2) that an integer n exists so that $R^n = R^{n+k}$ for each k. The following argument is a slight modification of one used by the writer in [2]. By putting $A = R^n$ one has $A = A^j$ for each j. Now suppose that $A \neq 0$, and define a_1 so that $a_1 \in A$ and $Aa_1A \neq 0$ (this is possible, since $0 \neq A = A^3$). Write $Aa_1A = Aa_1A \cdot A^3$ and define a_2 so that $Aa_1A \cdot Aa_2A \neq 0$. Continuing this process one obtains (by induction) an infinite sequence of two-sided ideals $Aa_1A, Aa_2A, Aa_3A, \cdots$ so that $a_i \in A$ and $Aa_1A \cdot Aa_2A \cdots Aa_kA \neq 0$ for each k. By writing $Aa_iA = A_i$ and considering that $A_i \subseteq R$, it follows from $A_1 \supseteq A_1A_2 \supseteq A_1A_2A_3 \supseteq \cdots$ by condition (2) that an integer m exists so that $A_1 \cdot A_2 \cdots A_m = A_1 \cdot A_2 \cdots A_m \cdot A_{m+1}$. By successive right-hand multiplication it follows that $A_1 \cdot A_2 \cdots A_m = A_1 \cdot A_2 \cdots A_m \cdot A_{m+1}^k$ for each k; hence $A_{m+1}^k \neq 0$ for each k. On the other hand, since $A_{m+1} = Aa_{m+1}A$, where $a_{m+1} \in R$, it follows that a_{m+1} is contained in a certain nilpotent ideal, which implies that A_{m+1} is nilpotent. This contradiction is a consequence of the assumption that $A \neq 0$; hence $A = R^n = 0$.

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