BOUNDEDNESS OF ORTHONORMAL POLYNOMIALS ON LOCI OF THE SECOND DEGREE

BY DUNHAM JACKSON

1. **Introduction.** The consideration of polynomials in two real variables orthogonal with respect to integration along an algebraic curve gives rise to systems of orthogonal functions which constitute a generalization of well-known orthogonal systems of fundamental importance, with preservation of a number of their characteristic properties. (See [4]; the hypotheses to be admitted here with regard to the domain of orthogonality are in some respects more general than those explicitly formulated in the paper to which reference is made, but it will be clear that the differences have no effect on the formal properties of the orthogonal systems with which the argument is concerned.)

If the domain of integration is a locus of degree N, the system of orthogonal polynomials in x and y contains N polynomials of the n-th degree for each value of $n \geq N$, and $n + 1$ polynomials of the n-th degree for $n < N$. The corresponding functions of arc length can be used for the formal expansion in series of an "arbitrary" function defined on the curve. They satisfy a relation in the form of a Christoffel-Darboux identity, leading to a representation of the partial sum of the series which serves as basis for a theory of convergence. As in the case of orthogonal polynomials in a single variable, an elementary treatment of convergence is greatly facilitated if the normalized functions of the orthogonal system are bounded as n becomes infinite.

The domain of integration for the definition of orthogonality need not be the complete locus of an algebraic equation. For present purposes it is in fact always assumed to be finite in extent. Even if the entire locus is finite, an ellipse, for example, the integration may be extended only over an arc or set of arcs of it. Being algebraic, the curve is necessarily rectifiable. It will be understood that the variable of integration is arc length when the contrary is not expressly indicated. The definition may, however, involve a weight function of a wide degree of generality; and the consideration of a variety of weight functions is an essential feature of the present discussion.

For a straight line segment, with unit weight function, the orthogonal polynomials of the present theory are essentially Legendre polynomials in a single variable. For the unit circle with parametric representation $x = \cos \theta$, $y = \sin \theta$, and with unit weight function, they may be taken as polynomials which reduce on the circle to the sines and cosines of multiples of θ, and the corresponding series development of an arbitrary function is an ordinary Fourier series. For a line segment with a general weight function the construction leads

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