AN INVERSION FORMULA FOR THE STIELTJES TRANSFORM

By Harry Pollard

The present treatment of the Stieltjes transform is intended to parallel a recent study of the Laplace integral made by Boas and Widder [2]. The underlying idea, suggested by [5; 41–43], is to consider the *iterate* of the transform and to apply known methods to this.

Consider, for example, the special Stieltjes transform

$$f(u) = \int_0^\infty \frac{\varphi(t)}{u+t}\,dt. \tag{1}$$

An examination of its (formal) iterate

$$g(x) = \int_0^\infty \frac{f(u)\,du}{x+u} = \int_0^\infty \frac{du}{x+u}\int_0^\infty \frac{\varphi(t)}{u+t}\,dt$$

suggests that $\varphi(t)$ can be determined by applying a known inversion for the iterated Stieltjes transform to $g(x)$, see [3; 18]. We obtain in this way the result

$$\varphi(t) = \lim_{k\to\infty} e_k t^{k-1}\int_0^\infty u^k \frac{\partial^{2k-1}}{\partial u^{2k-1}}\left\{\frac{u^{2k-1}}{(u+t)^{2k}}\right\} f(u)\,du, \tag{2}$$

where the $\{e_k\}$ are suitable constants.

Note that the formula (2) requires a knowledge of $f(x)$ on the real axis. Inversion formulas previously given involve $f(x)$ for complex values of the argument, or require a knowledge of the derivatives of $f(x)$. For these and related material we refer the reader to Widder [8; Chapter VIII].

By the same device we obtain new criteria for the representation of a function in the form

$$f(x) = \int_0^\infty \frac{d\alpha(t)}{x+t} \tag{3}$$

with $\alpha(t)$ of preassigned type. For example, $f(x)$ has the representation (3) with non-decreasing $\alpha(t)$ if and only if $f(x)$ is continuous for $x > 0$, $\lim_{x\to\infty} f(x) = 0$, and

$$(-1)^k \int_0^\infty u^k \frac{\partial^{2k-1}}{\partial u^{2k-1}}\left\{\frac{u^{2k-1}}{(u+x)^{2k}}\right\} f(u)\,du \geq 0 \qquad (x > 0;\ k = 2, 3, \cdots).$$

We also obtain for the first time a necessary and sufficient condition for the representation of a function $f(x)$ in the most general convergent Stieltjes transform (3).

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