CONVERGENCE, CLOSURE AND NEIGHBORHOODS

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It is well known that if a closure is defined in a set X satisfying the axioms of Riesz-Kuratowski, then a neighborhood system can be defined subject to certain corresponding axioms such that this neighborhood system induces the given closure; the converse process is also possible except that it may lead back to an equivalent neighborhood system but not to the original. G. Birkhoff [3] extended this equivalence of properly restricted neighborhood and closure topologies to include the equivalence of any Hausdorff space X with a space in which the topology is defined by assigning limits to functions defined from directed systems to X. Tukey [9], [10] extended this by relaxing the conditions imposed on closure, neighborhood and limit definitions retaining additivity of closure and using only functions on directed systems. Later I [5] showed that the monotone closures are just those which can be defined from neighborhood systems and from convergence of functions on sufficiently general ordered systems. Independently Ribiero [8] showed that every neighborhood space of a certain type could be defined in terms of functions on the same sort of ordered systems [see §4].

In all these cases certain more or less arbitrary restrictions are put on a given definition by closure, neighborhoods or convergence of a topology in X and then an attempt is made to induce such a topology by means of one of the other definitions. It seems that this question could as well be asked from the other end. In a space such as Euclidean n-space there are simple methods of defining any one of these three in terms of any other so that it is possible to go from any one through a finite chain of these transformations and return to essentially the original starting point. These transformations are still valid in far more general spaces [9]. The first problem of this paper is this: What properties of closure, neighborhoods, and convergence are necessary and sufficient for this equivalence to hold? It turns out, for example, that no assumption past maximality of neighborhood systems has any effect in this problem; all the usual restrictions are completely irrelevant to this fundamental question of equivalence. Practically all the conditions usually imposed on closure are also irrelevant; all that is needed is the property of *monotony*: If $\mathbf{x} \subset \mathbf{x}'$, then $c\mathbf{x} \subset c\mathbf{x}'$. The problem of finding proper conditions on convergence is not solved so neatly, but simple sufficient conditions are given. A second problem that arises in attempting to solve the first is to determine the possible variations in case the equivalence does not hold. A third problem is this: If we retain all allowable generality of, say, closure, how much restriction can be placed on a definition of neighborhoods or convergence that gives that closure?

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