SYSTEMS OF LINEAR EQUATIONS OF ANALYTIC TYPE

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1. **Introduction.** A linear problem concerning analytic functions is often expressed in terms of the power series coefficients of the functions; this leads the problem to a system of linear equations, in the coefficients, of the following form:

$$A_n[X] \equiv \sum_{k=0}^{\infty} a_{nk}x_k = c_n \qquad (n = 0, 1, \cdots), \tag{1.1}$$

where X represents the sequence $\{x_n\}$. We denote by $\mathcal{A}$ the system of linear forms appearing on the left side of (1.1).

Obvious examples of (1.1) come from linear differential and difference equations. An interesting special problem leading to a system (1.1) is the problem of Takenaka [1], [4], [5] wherein a sequence $\{a_n\}$ lying in $|z| \leq 1$ is given, and it is required to find the largest number r such that every function of exponential type less than r is identically zero if $f^{(n)}(a_n) = 0$ for all $n \geq 0$. If we set $f(z) = \sum_0^{\infty} x_n \cdot z^n/n!$, the conditions $f^{(n)}(a_n) = 0$ become

$$\sum_{k=0}^{\infty} (a_n^k/k!)x_{n+k} = 0 \qquad (n = 0, 1, \cdots),$$

and this is of type (1.1). It has the further property of being of *triangular form*; that is, in the n-th equation no x_i of index less than n occurs.

The general triangular system can be written

$$A_n[X] \equiv \sum_{k=0}^{\infty} a_{n,n+k}x_{n+k} = c_n \qquad (n = 0, 1, \cdots). \tag{1.2}$$

A particular but important case of (1.2) was investigated by Perron [3] in extending a classical result of Poincaré on the asymptotic character of solutions of a linear recurrence equation.

Taking cognizance of the fact that the x_n's are power series coefficients, we introduce the

Definition. *By the type of the sequence* $\{x_n\}$ *is meant the number* $((x_n))$, *where*

$$((x_n)) \equiv \limsup_{n=\infty} |x_n|^{1/n}. \tag{1.3}$$

(The notation $((x_n))$ is less cumbersome than the "lim sup", especially when the fractional exponent is considered.)

Two important problems concerning system (1.1) are these: (i) To determine the range of validity of the transformation from $\{x_n\}$ to $\{c_n\}$. (ii) To determine,

Received September 3, 1943; presented to the American Mathematical Society December 28, 1942.