ALGEBRAIC-LOGARITHMIC SINGULARITIES AND HADAMARD'S DETERMINANTS

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1. If $s = u + iv$ is a complex constant, k a non-negative integer, and $\varphi(z)$ a function which is holomorphic and not zero at z_0, and if $\log(z - z_0)$ is assigned its principal value in the region $|z| > |z_0|$, then

$$g(z) = e^{-s \log (z-z_0)}[\log (z - z_0)]^k \varphi(z)$$

will be said to be an algebraic-logarithmically singular element at z_0. The element will be said to have compound order

$$(u, k) \text{ if } s \neq 0, -1, -2, \cdots,$$

$$(u, k - 1) \text{ if } s = 0, -1, -2, \cdots, \text{ and } k > 0,$$

$$(-\infty, 0) \text{ if } s = 0, -1, -2, \cdots, \text{ and } k = 0.$$

Of two compound orders (a, b) and (a', b') the first will be defined to be greater than the second if either $a > a'$ or $a = a'$ and $b > b'$. A function $f(z)$ will be said to have an algebraic-logarithmic singularity at z_0 if $f(z)$ is the sum of a finite number of algebraic-logarithmically singular elements at z_0. The compound order of this singularity (or of $f(z)$ at z_0) will be the greatest of all the compound orders of the singular elements at z_0; and the algebraic-logarithmic singularity will be said to be ordinary if only one of the singular elements at z_0 has the compound order of the singularity at z_0.

It is the purpose of this paper to present the solution of the following problem: Given a sequence $\{a_n\}$, and given that the function $f(z)$ represented by the series $\sum a_n/z^n$ has an ordinary algebraic-logarithmic singularity at z_0 and is holomorphic in the region $|z| > |z_0|$ save for poles $z_1, z_2, \cdots, z_m$ of multiplicities $r_1, r_2, \cdots, r_m$, to find the compound order of $f(z)$ at z_0.

In §§3 and 4 we shall prove a theorem that covers a special case of the problem. A simple extension provides the general solution stated in §5.

2. Our solution depends on a result due to Jungen [1] and on the evaluation of Hadamard's determinants

$$D_{n,p} = \begin{vmatrix} a_n & a_{n+1} & \cdots & a_{n+p} \\ a_{n+1} & a_{n+2} & \cdots & a_{n+p+1} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n+p} & a_{n+p+1} & \cdots & a_{n+2p} \end{vmatrix}$$

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