CARATHEODORY AND GILLESPIE LINEAR MEASURE

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1. Carathéodory [3] in 1914 defined a linear measure which has since come to be named Carathéodory linear measure. In 1940, A. P. Morse and J. F. Randolph published jointly a paper [4] introducing Gillespie linear measure. In this paper we shall be interested in a known inequality involving these two linear measure functions. It will be shown that the particular relation in question is actually the best possible, the word "best" being used in a sense which will be subsequently made clear.

We shall first define the two measure functions and then mention a few necessary properties.

Let U be a bounded convex set of diameter-length $d(U)$. If the set U is a line segment, let $c(U)$ denote twice the length of this segment. Otherwise $c(U)$ will represent the ordinary length of the simple closed curve bounding the set U. A bounded convex set has a boundary which may be defined parametrically by two continuous functions of bounded variation. It follows that the boundary of such a set is a simple closed curve having ordinary length.

Let A be a plane point set. For each sequence of bounded convex open sets $U_1, U_2, \cdots$ lying in the plane and satisfying the conditions

(a) the union $\sum_i U_i$ includes the set A as a subset,

(b) $d(U_i) \leq \rho$, $i = 1, 2, \cdots$, where ρ is a pre-assigned positive number, form the sums $\sum d(U_i)$ and $\sum_i \frac{1}{2}c(U_i)$.

Carathéodory in his definition of bounded convex open sets did not even require that the sets $U_1, U_2, \cdots$ be convex, but showed if such were the case that $L^*(A)$ remained unchanged. Let $L_\rho(A)$ and $G_\rho(A)$ denote respectively the greatest lower bound (finite or positively infinite) of all such sums $\sum_i d(U_i)$ and $\sum_i \frac{1}{2}c(U_i)$. If ρ decreases, clearly $L_\rho(A)$ and $G_\rho(A)$ cannot decrease.

Define $\lim_{\rho \to 0} L_\rho A = L^*(A)$ and $\lim_{\rho \to 0} G_\rho(A) = G^*(A)$. $L^*(A)$ and $G^*(A)$ are called respectively the outer Carathéodory and Gillespie linear measures of the set A. These two measure functions both satisfy the six fundamental measure postulates as formulated by Carathéodory [3], [4]. Clearly it follows that closed sets, open sets F_σ's, G_δ's, etc. are all measurable. Only measurable sets will be used throughout. We shall adopt the customary notation $L(A)$ and $G(A)$ to represent the corresponding measures of a measurable set A. If γ is a simple rectifiable curve having ordinary length l and if $[\gamma]$ is the set of points lying on such a curve, it may be shown that the set $[\gamma]$ is measurable, having both its Carathéodory and Gillespie linear measure equal to the correct length of the curve γ, i.e., $L([\gamma]) = G([\gamma]) = l$, see [3], [4].

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