A REMARKABLE CLASS OF ALGEBRAIC INTEGERS. PROOF OF A CONJECTURE OF VIJAYARAGHAVAN

By R. Salem

1. Let θ be a number greater than 1, λ a positive number, and let $(\lambda\theta^n)$ be the fractional part of $\lambda\theta^n$ $(n = 1, 2, \cdots)$. The distribution of the values of $(\lambda\theta^n)$ is of importance in a number of problems and has been attracting attention for a long time. In 1935, Koksma [1] established that, for a given λ, the sequence of numbers $(\lambda\theta^n)$ $(n = 1, 2, \cdots)$ is uniformly distributed in the interval $(0, 1)$, when θ does not belong to an exceptional set of measure zero (depending on λ). In 1938, Pisot [2] studied the exceptional values of θ; he considered especially the values of θ such that, for some λ, $(\lambda\theta^n)$ has no limit points other than 0 or 1, and more particularly the values of θ for which there exists a $\lambda \neq 0$ such that the series $\sum_0^\infty \sin^2 \pi\lambda\theta^n$ converges. He proved the following outstanding result: if θ is such that there exists a $\lambda \neq 0$ such that the series $\sum_0^\infty \sin^2 \pi\lambda\theta^n$ converges, then θ is an algebraic integer whose conjugates have their moduli all less than 1, and λ is an algebraic number of the field $K(\theta)$; conversely, if θ is an algebraic integer of the above described type, there exist values of λ such that $\sum_0^\infty \sin^2 \pi\lambda\theta^n < \infty$ and one can take, in particular, $\lambda = 1$.

The class of algebraic integers defined above has been studied independently in 1940 by Vijayaraghavan [4]. In 1943, the author [3] has proved that a symmetrical perfect set of the Cantor type and of constant ratio of dissection ξ is a set of uniqueness for trigonometrical series if and only if ξ is the reciprocal of an algebraic integer of the above class and has proposed to call these algebraic integers: "Pisot-Vijayaraghavan numbers".

The problem of the distribution of the Pisot-Vijayaraghavan numbers in the interval $(1, \infty)$ seems to have remained open and the purpose of this paper is to give a solution of this problem.

In his paper quoted above, Vijayaraghavan states: "It seems to me to be improbable that the set S (the set of Pisot-Vijayaraghavan numbers) could be dense everywhere in some interval, or be dense in itself". We shall prove that the double conjecture of Vijayaraghavan is true. In fact we shall prove that the set of Pisot-Vijayaraghavan numbers is a closed set. Since it is denumerable it follows that it is: (1) nowhere dense; (2) not dense in itself; (3) reducible, i.e., all its derived sets after a certain rank (finite or transfinite) are empty.

The proof of our theorem is based on the following lemma:

2. Lemma. *If we are given a Pisot-Vijayaraghavan number θ, there always exists*

Received December 27, 1943.