AN EXPRESSION CONNECTED WITH THE AREA OF A SURFACE $z = F(x, y)$

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1. **Introduction.** In the chapter on "area" in Saks's book [1], the concluding section contains an error pointed out by Jarnik, which renders invalid a theorem suggested by me. The second part of Saks's proof of [1; 182, Theorem (8.3)] is based on the false inequality (near the bottom of p. 183)

$$R_n(x, y; \alpha, \beta) \leq \frac{1}{k} \sum_{j=0}^{k-1} R_n(x + j\alpha/k, y + j\beta/k; \alpha/k, \beta/k).$$

The existence of an error was also pointed out by T. Radó and P. V. Reichelderfer. The error was not mine, but I am partly to blame for suggesting that the theorem could easily be proved in this sort of way, and I failed to detect the error during proof reading.

It is easily seen that the correct part of Saks's proof establishes only the following result:

(1.1) *The area $S(F; I_0)$ of a continuous surface $z = F(x, y)$ on an interval I_0 of x, y does not exceed the lower limit, as α and β tend to* 0 *(by non-zero values), of the expression*

$$S^*_{\alpha\beta}(F; I_0) = \iint_{I_0} \left\{ \left[\frac{F(x + \alpha, y) - F(x, y)}{\alpha} \right]^2 + \left[\frac{F(x, y + \beta) - F(x, y)}{\beta} \right]^2 + 1 \right\}^{\frac{1}{2}} dx\, dy$$

and in order that the function F be of bounded variation on I_0 (in the Tonelli sense), it is necessary and sufficient that

$$\limsup_{\alpha, \beta \to 0} \frac{1}{| \alpha | + | \beta |} \iint_{I_0} | F(x + \alpha, y + \beta) - F(x, y) | \, dx\, dy < \infty.$$

Strictly the necessity requires further proof; this is easily supplied by minor modifications of ideas used by Saks, or it may be derived from our Theorem (2.7) below as it is not used in establishing the latter.

The object of this note is to show that there exist surfaces for which the lower limit of $S^*_{\alpha\beta}$ exceeds S, and to obtain simple expressions for the upper limit of $S^*_{\alpha\beta}$ and for S, by means of which it is possible to state exactly under what circumstances S is the unique limit of $S^*_{\alpha\beta}$.

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