HOMOTOPY REDUCTIONS OF MAPPINGS INTO THE CIRCLE

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1. **Introduction.** In this paper it will be shown that an arbitrary continuous mapping of a locally connected continuum into the circle is reducible by homotopy to one which is the resultant of a monotone mapping followed by a light interior one. Since the action of these two types of transformations on the structure of a set is greatly simplified and can be fairly well analyzed [4] and since the homotopy or "deforming" operation is known to leave unaffected the essential topological action of a transformation, we are thus able to bring the entire important class of continuous mappings of locally connected continua into the circle within the range of effective study and analysis.

A continuous mapping $f(x)$ of a continuum A onto a set B is (a) *monotone* if the inverse of each point of B is connected, (b) *interior* (or *open*) if the image of every open set in A is open in B, (c) *light* if the inverse of each point of B is totally disconnected, (d) *quasi-monotone* if, for each continuum K in B with a non-empty interior, $f^{-1}(K)$ has just a finite number of components and each of these maps onto K under f. As shown by [3] the quasi-monotone mappings on locally connected continua A turn out to be exactly those mappings which factor into the form $f_2 f_1(A)$, where f_1 is monotone and f_2 is light and interior. Hence our main result admits the concise formulation: Any continuous mapping of a locally connected continuum A into the circle S is homotopic to a quasi-monotone mapping. Thus each homotopy class of mappings of A into S contains a quasi-monotone mapping of A into S.

It will be convenient to consider the circle S into which our mappings operate as the unit circle $|z| = 1$ in the complex plane. Arithmetic operations then have meaning in S in the ordinary sense of the complex number operations. Thus we can multiply and divide mappings into S (see [1], [2] or [4]). We employ also the usual distance $\rho(f, g)$ between transformations f and g as a metric in the space S^A of all mappings of A into S. All spaces are assumed to be metric and it is understood that a continuum is a compact connected set.

The following well-known simple results will be of use to us and are included for the sake of completeness.

(1.1) *Any continuous mapping of a metric set A onto a proper subset of S is homotopic to a constant.*

For there is some value $a \in S$ such that $f(A)$ does not contain $-a$, and there is no loss in generality in supposing $a = 1$. Then the family

$$g(x, t) = \frac{t + (1 - t)f(x)}{|t + (1 - t)f(x)|} \qquad (x \in A, 0 \leq t \leq 1)$$

deforms $f(x) = g(x, 0)$ continuously into the mapping $g(x, 1) \equiv 1$ of A into 1.

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