FUNCTIONS OF EXPONENTIAL TYPE, II

BY R. P. BOAS, JR.

1. This note presents some new information on Whittaker's problem [10; 45], [7], [1]: to find the least upper bound W of numbers c such that if $f(z)$ is entire and of exponential type c, and if the unit circle contains at least one zero for $f(z)$ and each of its derivatives, then $f(z) \equiv 0$. It was previously known [8], [10], [1] that $.693 = \log 2 \leq W \leq \frac{1}{4}\pi = .785$. Here I shall show that $.718 < W < .748$. The method by which the new upper bound for W is obtained is due to G. Pólya. Pólya has also shown (unpublished result) that $W > \log 2$, but his proof yields no numerical lower bound greater than log 2. More careful computation would narrow the allowable range for W, but the methods of this note do not seem capable of determining W exactly. There is as yet no evidence either for or against the conjecture [5] that $W = 2/e = .736$.

I am indebted to Professor Pólya for suggesting that I should publish here his proof that $W < \frac{1}{4}\pi$ (§3), and to my wife for suggestions which have led to improvements in both numerical bounds for W.

2. We first obtain a lower bound for W. Let $\{\alpha_n\}$ be a sequence of complex numbers such that $|\alpha_n| \leq 1$. Let

$$g_n(z) = z^n(1 + h_n(z)) = z^n e^{\alpha_n z} - \alpha_n z^{n+1} e^{\alpha_{n+1} z}.$$

Let s denote the radius of the largest circle in which every analytic function can be expanded in a series of the functions $g_n(z)$, with uniform convergence in any concentric smaller circle. We have $W \geq s$. For, if $f(z)$ is of type $s' < s$, and C is a circle with center at the origin and radius between s' and s, then [4; 580ff]

$$f(z) = \int_C e^{zw} \varphi(w)\, dw, \tag{2.1}$$

where $\varphi(w)$ is analytic in $|w| > s'$. If we develop e^{zw}, as a function of w, in terms of the functions $g_n(w)$, substitute the expansion into (2.1), and integrate term by term, we obtain

$$f(z) = \sum_{n=0}^{\infty} c_n(z)[f^{(n)}(\alpha_n) - \alpha_n f^{(n+1)}(\alpha_{n+1})].$$

If $f^{(n)}(\alpha_n) = 0$ for $n = 0, 1, 2, \cdots$, it follows that $f(z) \equiv 0$.

We must therefore show that $s > .718$. To do this, it is enough [1] to obtain a function $h(z)$ which is a common majorant for all the $h_n(z)$ and for which $h(.718) < 1$.

Received December 1, 1943.