UNENDING CHESS, SYMBOLIC DYNAMICS AND A PROBLEM IN SEMIGROUPS

By Marston Morse and Gustav A. Hedlund

1. **Introduction.** The question of the possibility of an unending game of chess (with modified rules concerning a draw) leads to the problem of constructing an unending sequence of symbols from a given finite set so that the infinite sequence does not contain a block of the form EEe, where E is itself a block and e is the first symbol in E. Each block is supposed composed of a subsequence of *successive* symbols of the unending sequence. It was discovered [3] that an unending sequence generated from two symbols as previously defined and used to establish the existence of non-periodic recurrent motions in dynamics (cf. [1], [2]) has the desired property. The proof of this is published here for the first time.

The conventional rules governing chess contain rules concerning a draw by virtue of which every game contains a finite number of moves. Continental chess players have speculated upon various modifications of these rules and the possibility of an unending game of chess resulting therefrom. One such set of rules governing a draw was brought to the attention of Morse a number of years ago by members of the mathematics faculty of the University of Muenster. The set of pawns and pieces on the board prior to a move constitute what may be called a configuration. The *identity of two configurations* is understood to be complete identity both as to position and character of pieces and pawns. With this understood we see that there is at most a finite number of different configurations. These configurations are put together in a game in a sequence. In the terminology of the first paragraph a draw results if the sequence defined by the game contains a block of the type EEe. The conventional rules requiring that a pawn be moved or a piece taken after a certain number of moves are dropped in this modified game. The problem of the unending game under these modified rules had apparently been studied for some time in Germany without a solution being obtained.

It has also been pointed out to us by R. P. Dilworth that the existence of unending sequences of the type which yield a solution of the chess problem makes possible the construction of useful examples of semigroups. A semigroup is a set S of elements $a, b, c, \cdots$ in which an associative operation ab is defined. The element z is a zero element if $za = az = z$ for all a in S. If A and B are subsets of S, let AB denote the set of all products ab, where a and b are arbitrary elements of A and B, respectively. The semigroup S is said to be nilpotent if there exists an integer k such that $S^k = z$. It is possible to construct a non-nilpotent semigroup S generated by three elements, such that the square of

Received November 20, 1943.