SOME STUDIES ON CLOSURE RELATIONS

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The present paper is concerned with some investigations on the basic properties of closure relations. Such relations define topological spaces of a very general nature. It should be observed in particular that the additivity of the closure operation is not presupposed. To consider spaces of this generality appears desirable for a number of reasons. First, the closure operations occurring in many important mathematical theories are of this general nature. In algebraic systems the subsystems closed under the algebraic operations, for instance the subgroups of a group, define topologies of this kind. A second reason is that when the topologies are conceived of in this general manner the theory takes a more completed form which could otherwise not be achieved. Furthermore, the theory of topological spaces and the theory of structures become identified as different aspects of the same theory.

The point of departure is the fact that in every closure relation the closed sets form a complete structure and every complete structure can be obtained in this way. We introduce classes of structure equivalent closure relations, each class consisting of all closure relations whose structure of closed sets is isomorphic to a given complete structure Σ. It is shown that all closure relations in a class are obtainable by a so-called distinct (or basis) relativisation of the structure Σ with respect to its associated closure relation. One finds that in each class there can be at most one closure relation in which every point is closed. Similarly for a distributive structure there can be at most one $\mathfrak{T}$-space with an additive closure relation in the class. To any class there exists a dual class consisting of all closure relations whose structures of closed sets are isomorphic to the dual structure Σ^* of the given Σ. The structures of closed sets of all relative spaces of the spaces in a class are also determined. It is shown that they belong to those complete structures which are substructures of Σ with respect to union. The last sections deal with the representation of the given space as a relative space. One can prove that every space is a relative space of a self-dual space. Furthermore, one can show that every space is a relative space of some space with bicompactness properties of a very strong type.

Among the general references to investigations related to the present paper one should mention particularly G. Birkhoff [2], Wallman [10], Frink [4] and Ore [7].

1. **Basic properties of closure relations.** We shall begin by stating some of the basic properties of closure relations. See [7]. We define a *closure relation* Γ in a set S to be a correspondence which associates with every subset A of S

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