A GALOIS THEORY FOR DIFFERENTIAL FIELDS

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1. **Introduction.** In seeking to develop a Galois theory for simple algebraically transcendental extensions of partial differential fields, one is faced with two restrictions which force a deviation from the conventional algebraic approach. The concept of a minimal equation of an extension is lacking as is the notion of a normal field. Kolchin, in a recent paper [3], has resolved the first difficulty by associating with any zero of a differential polynomial a prime differential ideal. One finds that this ideal may be handled as effectively as the minimal polynomial in the algebraic instance.

A deeper restriction is the absence in general of a normal extension, an absence due to the existence of infinitely many conjugate extensions. Hence the Galois theory we shall develop can be neither the study of the subfields of a normal field nor an investigation of its automorphisms. It is true that isomorphisms of our fields, grouped into appropriate sets, play a major rôle in our presentation, but the resulting Galois theory has a more fundamental characterization. Although there are infinitely many elements in an extension, there are but a finite number of subfields containing the base field. We shall strive to exhibit a finite algebraic system, the lattice of whose subsystems is inversely isomorphic (union corresponding to crosscut) to the lattice of our subfields.

The finite algebraic system we shall produce is a special sort of multigroup, i.e., a group-like system with non-unique multiplication. In §2 we shall abstract from the literature a description of such properties of multigroups as we shall have occasion to use. We next recast conventional Galois theory into the form susceptible of generalization to differential fields, thereby pointing the analogy of our development to the algebraic case. In §4 the terminology and pertinent facts of differential fields are recalled. We follow this in §5 with a few additional theorems on differential fields. Theorem 1 in this section is the basic theorem of the paper; it is later used to show that the isomorphisms of an extension may be grouped into a finite number of sets. These sets, in §6, become the elements of our Galois multigroups. The usual Galois theorems are then shown to apply.

2. **Multigroups.** Dresher and Ore [1] have subjected multigroups to a systematic examination. We here reproduce some of the concepts they introduced and results they obtained.

A *multigroup* $\mathfrak{M}$ is an algebraic system with a single operation called multiplication subject to the following axioms.

I. If m_i , m_j are any two elements of $\mathfrak{M}$, their product is a subset of $\mathfrak{M}$:

$$m_i m_j = \{m_k\}.$$

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