AN ABSTRACT THEORY OF THE JORDAN-HÖLDER COMPOSITION SERIES

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The Jordan-Hölder theorem as developed in the theory of groups has a doubly two-fold aspect:

(1)	(2)
Each subgroup of a chain is normal under the original group.	Each subgroup of a chain is normal under the preceding subgroup of the chain.

(a) Two chains of permissible subgroups which admit no refinements contain the same number of elements.

(b) The set of all quotient groups (distinct or not) which occur between successive subgroups in one of the chains is exactly the set of all those arising between adjacent subgroups in the other chain.

An effectual transfer of (1), (a) to pure lattice theory has been accomplished, and some attention has been given to (2), (a). However, inasmuch as lattice-isomorphism between the lattices of the normal subgroups does not imply group-isomorphism of the quotient groups involved, (1), (b) and (2), (b) do not follow from lattice theory, i.e., considerations which are fundamentally group-theoretic must be introduced. We shall bridge this gap by dealing with an *operator lattice*, but shall restrict ourselves to (2), (a) and (2), (b).

Many theorems concerning the Jordan-Hölder situation have been established in various generalizations of the group concept. The similarities of the proofs indicate that they are embodied in an analogous theorem regarding a weaker algebra. In order to amalgamate the results of the various authors, it is necessary to weaken the properties of the group multiplication and postulate the characteristics of the "normal subgroups" more precisely.

Quasi-groups ("groups" with weakened or non-existent associative laws) constitute one type of generalized groups. An important subclass of quasi-groups are those of Hausmann and Ore [5]. Instead of the somewhat specialized associative laws employed by them, we shall utilize a weaker one suitable for arbitrary quasi-groups originated by R. J. Duffin [3] in an as yet unpublished investigation. Our normality law is essentially that of Murdoch [9], [10], [11].

Our paper also has an interpretation in terms of multigroups, in which two elements combine into a subset rather than a single element of the set. This notion began with the hypergroups of Marty [8] and Wall [14], but a Jordan-Hölder theorem was first proved by J. Kuntzmann [7] and in the most general

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