GENERALIZED DYNAMICAL TRAJECTORIES IN SPACE

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1. **Introduction.** The differential geometry of positional fields of force has been developed in [4], [6], [7], [9], [10]. The fields of force, considered heretofore, depend upon the position of the point. In our preceding work [11], we began the study of *generalized* fields of force in the plane which depend not only upon the position of the point but also upon the direction through that point. In this paper, we propose to develop the geometry of the dynamical trajectories of such generalized fields of force in space.

In a generalized field of force, there are ∞^5 dynamical trajectories. These are represented by a system of differential equations of the form

$$\text{(G)} \qquad z'' = K(x, y, z, y', z')y'', \qquad y''' = G(x, y, z, y', z')y'' + H(x, y, z, y', z')y''^2.$$

(In this paper, primes denote differentiation with respect to x.) Conversely, any system of curves of this type (G) may represent the dynamical trajectories of $\infty^{f(4)}$ generalized fields of force (the symbol $\infty^{f(4)}$ means that the force vector contains an arbitrary function of four variables [8]). Any such system of curves may be considered to be a generalization to three dimensions of the dynamical trajectories in a generalized field of force in the plane. Since these plane generalized dynamical trajectories were said to be of the type (G), we shall find it appropriate to term our new system of ∞^5 trajectories as being of *the three-dimensional type* (G).

Of course, not every such system of curves represents the trajectories of a positional field of force. Kasner proved that the family of ordinary trajectories is characterized in space by a set of *four* independent geometric properties (and in the plane by a set of *five* independent geometric properties). Of these, Properties I and II are equivalent analytically to stating that a system of ordinary dynamical trajectories in space must be represented by a system of two differential equations of the three-dimensional type (G). The other geometric properties specialize the functions K, G and H.

Properties I and II may be described as follows. For each of the ∞^1 curves of the ∞^5 trajectories which pass through a given lineal element E, construct the osculating plane and the osculating sphere at E. Then our two properties are

(I) The ∞^1 trajectories all have the same osculating plane.

(II) The locus of the centers of the osculating spheres is a straight line.

It is found that each of the ∞^3 associated plane systems S of a system of ∞^5 generalized dynamical trajectories is a system of ∞^3 generalized dynamical trajectories.

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