DECOMPOSITION OF ADDITIVE SET FUNCTIONS

By C. E. Rickart

1. **Introduction.** This paper is concerned with the decomposition of additive set functions defined on a σ-field $\mathfrak{M}$ to a linear normed vector space $\mathfrak{X}$. The principal results are contained in Theorems 3.3, 3.8 and 4.5. Special cases of Theorems 3.3 and 3.8 were previously obtained by R. S. Phillips [7]. Theorem 4.5 is a generalization of the familiar Lebesgue decomposition theorem [9; 35] to functions whose values lie in a linear normed space. In this generalization, we are also able to replace the measure function by an arbitrary outer measure.

Throughout the discussion, M will stand for an arbitrary abstract set of points. A class $\mathfrak{R}$ of subsets of M is called a "ring" provided it contains the union and intersection of any pair of its elements. If a ring also contains the complement in M of each of its elements, then it is called a "field". A ring which contains the union and intersection of any denumerable sequence of its elements is called a "σ-ring". Similarly, a σ-ring which is a field is a "σ-field". A subclass $\mathfrak{A}$ of a ring $\mathfrak{R}$ is said to be "hereditary in $\mathfrak{R}$" provided it contains the intersection of each of its elements with every element of $\mathfrak{R}$. A ring $\mathfrak{A}$ which is hereditary in a larger ring $\mathfrak{R}$ is called an "ideal of $\mathfrak{R}$". We will be concerned throughout with only one σ-field (of subsets of M) which will be denoted by $\mathfrak{M}$. All sets considered will be assumed, without exception, to be elements of $\mathfrak{M}$. If a class of sets is hereditary in $\mathfrak{M}$, then it will simply be called "hereditary", and if it is an ideal of $\mathfrak{M}$, then it will simply be called an "ideal". Most of the above definitions will be found in [2; 1, 58].

The union and intersection of two sets e_1, e_2, the union and intersection of a denumerable sequence of sets $\{e_n\}$, and the complement of a set e will be denoted respectively by the symbols $e_1 \cup e_2$, $e_1 \cap e_2$, $\sum e_n$, $\prod e_n$ and Ce. The usual variations on these symbols will also be used. The small Greek letter π will always stand for a *finite* set of positive integers or other indices. The notation $\{x \mid P\}$ will be used to indicate the class of all elements x which satisfy a given property P.

2. **Strong boundedness of set functions.** We will be concerned in what follows with functions $x(e)$ defined in the σ-field $\mathfrak{M}$ to a linear normed space $\mathfrak{X}$ [1; 53]. The domain $\mathfrak{D}$ of $x(e)$ throughout the discussion will be assumed to be hereditary. $x(e)$ is said to be "additive" provided $x(e_1 \cup e_2) = x(e_1) + x(e_2)$, for disjoint e_1, e_2 such that $e_1 \cup e_2 \in \mathfrak{D}$. $x(e)$ is "completely additive" provided $x(\sum e_n) = \sum x(e_n)$, for any sequence of disjoint sets $\{e_n\}$ such that $\sum e_n \in \mathfrak{D}$. It is evident that the series will be unconditionally convergent [5].

Received May 13, 1943; presented to the American Mathematical Society February 27, 1943.