THE STABILITY OF SOLUTIONS OF LINEAR DIFFERENTIAL EQUATIONS

By Richard Bellman

1. **Introduction.** We want to consider differential equations of the form

$$y^{(n)}(x) + P_2(x)y^{(n-2)}(x) + \cdots + P_n(x)y(x) = 0 \quad (x \geq 0, n = 2, 3, \cdots) \tag{1.1}$$

and prove that under sufficiently small variations of the coefficients $P_k(x)$ the solutions, and their derivatives, will preserve boundedness. We shall estimate the variation between $P_k(x)$ and $Q_k(x)$ by means of

$$a_k(x) = \int_0^x | P_k(x) - Q_k(x) | \, dx. \tag{1.2}$$

Levinson [2] considered the case $n = 2$, with $Q_2(x) \equiv a$, and derived the following theorem:

If $x(t)$ satisfies the differential equation

$$x''(t) + \phi(t)x(t) = 0 \tag{1.3}$$

and

$$\alpha(t) = \int_0^t | \phi(t) - a | \, dt, \tag{1.4}$$

then, $a \neq 0$,

$$x(t) = O(\exp(\tfrac{1}{2}a^{-\frac{1}{2}}\alpha(t))). \tag{1.5}$$

In particular, if $\alpha(t)$ is bounded, $x(t)$ is bounded. He showed by example that this is a "best possible" result.

Cesari [1] proved the more general theorem:

If the differential equation

$$z^{(n)}(x) + a_1 z^{(n-1)}(x) + \cdots + a_n z(x) = 0, \tag{1.6}$$

where the a_k are constants, has only bounded solutions, and if

$$\lim_{x \to \infty} f_j(x) = a_j \qquad (j = 1, 2, \cdots, n), \tag{1.7}$$

$$\int_0^\infty | f_j(x) - a_j | \, dx < \infty, \tag{1.8}$$

then the differential equation

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