ALMOST PERIODIC GAP SERIES

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The purpose of this note is to prove the following analogue for a very general class of almost periodic functions of a well-known theorem of Fourier series, namely,

THEOREM. *If* $\sum_1^\infty a_k e^{i\lambda_k t}$, $\lambda_{k+1}/\lambda_k \geq \lambda > 1$, *is the Fourier expansion of a function* $f(t)$ *belonging to B a.p.*,

$$a_k = M(f(t)e^{-i\lambda_k t}) = \lim_{T\to\infty} \frac{1}{T} \int_{-T}^{T} f(t)e^{-i\lambda_k t}\, dt,$$

then the function $f(t)$ *necessarily belongs to* M^p, $p \geq 1$.

M^p denotes the class of functions such that

$$M^p(|\, f(t)\, |) = \lim_{T\to\infty} \left(\frac{1}{T} \int_0^T |\, f(t)\, |^p\, dt\right)^{1/p}$$

exists.

We will need the following lemmas:

LEMMA 1. *If* $\lambda_{k+1}/\lambda_k \geq \lambda > 1$, $P_n(t) = \sum_1^n a_k e^{i\lambda_k t}$,

$$M^p(P_n(t)) \leq A_{p,\lambda}\Big(\sum_1^n |\, a_k\, |^2\Big)^{\frac{1}{2}} \qquad (p \geq 1).$$

LEMMA 2. *Under the conditions of the preceding lemma,*

$$M^1(P_n(t)) \geq B_\lambda\Big(\sum_1^n |\, a_k\, |^2\Big)^{\frac{1}{2}},$$

where B_λ *depends only upon* λ.

The proofs of Lemmas 1 and 2 for the case of purely periodic functions will be found in [2; 216–218], and only a transcription of notation is needed to extend to the preceding cases.

LEMMA 3 [1; 104–109, §8]. *The Fourier series*

$$\sum_1^\infty a_k e^{i\lambda_k t}$$

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