CONFORMAL MINIMAL VARIETIES

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Let the functions

$$x^i = x^i(u^1, u^2, \cdots, u^n) \qquad (i = 1, 2, \cdots, m; m \geq n \geq 2), \tag{1}$$

defined for $(u^1, u^2, \cdots, u^n)$ in a domain D of Euclidean n-space, be coördinate functions of an n-dimensional variety V_n in Euclidean m-space.

The map of D on V_n given by (1) is conformal if and only if there exists a function $\lambda(u^1, u^2, \cdots, u^n)$ in D such that

$$\sum_{i=1}^{m} \frac{\partial x^i}{\partial u^j} \frac{\partial x^i}{\partial u^k} \equiv [\lambda(u^1, u^2, \cdots, u^n)]\delta_{j,k} \qquad (j, k = 1, 2, \cdots, n), \tag{2}$$

where $\delta_{j,k}$ is the Kronecker delta.

By analogy with the notion of a pair of conjugate harmonic functions in the theory of functions of a complex variable, if the functions (1) satisfy conditions (2) and

$$\sum_{j=1}^{n} \frac{\partial^2 x^i}{\partial u^{j2}} = 0 \qquad (i = 1, 2, \cdots, m),$$

then the functions (1) have been called [1], [3] *a set of conjugate harmonic functions* in D.

An extension of a theorem of Weierstrass gives the following geometric interpretation of the above definitions:

A necessary and sufficient condition that a V_n be a minimal variety in conformal representation is that the coördinate functions form a set of conjugate harmonic functions.

It is well known that any smooth V_2 can be mapped conformally on a domain in the plane and that the most general conformal map of a Euclidean space E_n on itself for $n \geq 3$ is the product of inversions with respect to hyperspheres, rigid motions, and transformations of similitude [2; 375–376]. A geometric characterization of spaces conformally equivalent to Euclidean n-space, for $n \geq 4$, has been given by Haantjes [4].

It follows that not necessarily can all minimal varieties V_n be represented conformally on E_n. Indeed, a simple proof has been given [1] that the only sets of conjugate harmonic functions, for $m = n \geq 3$, are sets of constants and linear functions; and it has been stated [1] that this result holds also for $m \geq n \geq$

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