THE COEFFICIENTS OF SCHLICHT FUNCTIONS

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1. Let $F(E)$ be the family of functions which are regular and schlicht in the interior of the unit circle E, and let $F(E)$ be normalized by the condition that at the origin each function has the value zero and its first derivative the value one. The exact upper bounds of the moduli of the second and third coefficients of functions of $F(E)$ are known to be 2 and 3 respectively, but exact upper bounds for the higher coefficients are not known, and the problem of determining them, although considered by many mathematicians in the last twenty-five years, is still one of the most famous outstanding problems of function theory. It has been widely conjectured from the beginning that the correct upper bound for the n-th coefficient is n despite the fact that at various times there has appeared to be some evidence against this belief. We denote by $P(E)$ the problem of determining the bounds of the coefficients of functions in $F(E)$.

Now $P(E)$ is only a special case of the following more general problem, and certain interesting aspects of $P(E)$ appear by considering it in its general setting. Let $\mathbf{C}$ be the set of domains into which the unit circle is mapped by functions of $F(E)$ and, if G is a domain of $\mathbf{C}$, let $F(G)$ be the family of functions $f(\zeta)$ which are regular and schlicht for $\zeta \subset G$ and which, near $\zeta = 0$, have the form

$$f(\zeta) = \zeta + a_2\zeta^2 + \cdots + a_n\zeta^n + \cdots.$$

If we write

$$\alpha_n = \alpha_n(G) = \sup_{f \subset F(G)} | a_n |,$$

each domain G of $\mathbf{C}$ determines a point $(\alpha_2, \alpha_3, \cdots)$. It is well known that $F(G)$ is a normal family and so, for each $n > 1$, there is at least one function f of $F(G)$ the n-th coefficient of which has absolute value α_n. The general problem is to determine the point $(\alpha_2, \alpha_3, \cdots)$ when G is given.

For each fixed $n > 1$ we write

$$\gamma_n = \inf \alpha_n(G) \qquad (G \subset \mathbf{C}),$$

$$\Gamma_n = \sup \alpha_n(G) \qquad (G \subset \mathbf{C}).$$

We shall see (§5) that $\alpha_n(E) = \gamma_n$ and that, if the conjecture $\alpha_n(E) = n$ is true, then $\Gamma_n = 4^{n-1}$ (but, given n, there are other domains G, depending on n, with $\alpha_n = \gamma_n$ although E is probably the only one with $\alpha_n = \gamma_n$ for all n). A proof of the conjecture for the unit circle would therefore yield exact upper and lower bounds for $\alpha_n(G)$, $G \subset \mathbf{C}$. It is easily shown that there exist domains G of $\mathbf{C}$ with $\alpha_n = \Gamma_n$.

We bring to the coefficient problem a new variational method which yields,

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