CONVERGENCE OF NON-HARMONIC FOURIER SERIES

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1. **Introduction.** M. Kac [1; 545] proved the following theorem:

If

$$\sum_{j \neq k} \frac{1}{(\lambda_j - \lambda_k)^2} < \infty,$$

and for some $\epsilon > 0$,

$$\sum_1^\infty | a_k |^{2-\epsilon} < \infty,$$

the series

$$\sum_1^\infty a_k e^{i\lambda_k x}$$

converges almost everywhere.

We wish to show that the theorem holds under less restrictive conditions, and that the method of proof can be simplified so as to include the theorem as a special case of a theorem of Zygmund [2].

2. Theorem. *The series*

$$\sum_1^\infty a_k e^{i\lambda_k x}$$

converges almost everywhere, provided the following conditions hold:

$$\sum_1^\infty | a_k |^p < \infty \qquad (0 < p < 2), \tag{1}$$

$$\lambda_{k+1} - \lambda_k \geq b > 0, \tag{2}$$

$$\lambda_1 > 0. \tag{3}$$

(*Condition* (3) *is no essential restriction.*)

Proof. Define a function $f(x)$ as follows:

$$f(x) = a_k \qquad (\lambda_k \leq x \leq \lambda_k + b),$$

$$f(x) = 0 \qquad \text{(all other } x).$$

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