A THEOREM OF FÉDOROFF AND BINNEY

By Maxwell Reade

The object of this note is to extend a theorem on harmonic functions, due to Fédoroff [2] and Binney [1], to subharmonic functions (for a list of properties of subharmonic functions, see [4]), and then, more interesting still, to exhibit the attendant mass distribution [4] for the subharmonic functions; the former will depend upon a theorem of Levin [3; 363–410], while the latter will depend upon the extension theorem of Reichelderfer and Ringenberg [5].

Theorem (Levin). *If the function $\Phi(x, y)$ is continuous with its partial derivatives of the first order in the unit circle $\mathfrak{D}: x^2 + y^2 \leq 1$, then a necessary and sufficient condition that $\Phi(x, y)$ be subharmonic in $\mathfrak{D}$ is that for each oriented square S in $\mathfrak{D}$,*

$$\int_S \Phi_x(x, y)\, dy - \Phi_y(x, y)\, dx \geq 0.$$

Extension Theorem (Reichelderfer and Ringenberg). *Let $\mathfrak{R}$ denote the class of all oriented closed rectangles in $\mathfrak{D}$. A necessary and sufficient condition that a given set function $\mu(R)$, defined on $\mathfrak{R}$, admit a completely additive extension to a closed range in $\mathfrak{D}$ is that it satisfy condition $\mathfrak{C}$: If $r_1, r_2, \cdots, r_n, \cdots$ is a finite or denumerable sequence of mutually exclusive rectangles in $\mathfrak{R}$, and if $R_1, R_2, \cdots, R_n, \cdots$ is any finite or denumerable sequence of rectangles in $\mathfrak{R}$ such that*

$$\sum_n r_n \subset \sum_n R_n ,$$

then

$$\sum_n \mu(r_n) \leq \sum_n \mu(R_n).$$

The theorem we intend to establish is the following.

Theorem. *If $u(x, y)$ and $v(x, y)$ are continuous in $\mathfrak{D}$, and if*

$$(1) \qquad \int_R u(x, y)\, dy - v(x, y)\, dx \geq 0, \qquad \int_R u(x, y)\, dx + v(x, y)\, dy = 0$$

hold for each oriented rectangle R in $\mathfrak{D}$, then there exists a function $\Phi(x, y)$, subharmonic in $\mathfrak{D}$, such that $\Phi_x(x, y) \equiv u(x, y)$, and $\Phi_y(x, y) \equiv v(x, y)$; the mass function associated with $\Phi(x, y)$ is given by the non-negative completely additive extension of the rectangle function

$$(2) \qquad \mu_1(R) \equiv \int_R u(x, y)\, dy - v(x, y)\, dx = \int_R \Phi_x(x, y)\, dy - \Phi_y(x, y)\, dx.$$

Received March 8, 1943.