THE FACTORIZATION OF RANK TENSORS

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Let $C^{(i)}_{(j)} \equiv C^{i_1 i_2 \cdots i_p}_{j_1 j_2 \cdots j_p}$ be an idempotent numerical tensor associated with an n-dimensional coördinate system; that is, $C^{(i)}_{(j)}$ is an absolute numerical tensor satisfying the relation

$$C^{(i)}_{(m)} \cdot C^{(m)}_{(j)} = C^{(i)}_{(j)} . \tag{1}$$

In connection with the algebra determined by $C^{(i)}_{(j)}$ there is the numerical invariant r_c, the rank of $C^{(i)}_{(j)}$, which in general we shall denote simply by r. It is defined [1], [2] as the greatest value of k for which the tensor

$$C^{(i_1)\cdots(i_k)}_{(j_1)\cdots(j_k)} = \begin{vmatrix} C^{(i_1)}_{(j_1)} & \cdots & C^{(i_1)}_{(j_k)} \\ \cdots & \cdots & \cdots \\ C^{(i_k)}_{(j_1)} & \cdots & C^{(i_k)}_{(j_k)} \end{vmatrix} \tag{2}$$

does not vanish. The rank tensor of $C^{(i)}_{(j)}$, by means of which the inverse and determinant of a tensor $A^{(i)}_{(j)}$ are formed, is

$$C^{(i_1)\cdots(i_r)}_{(j_1)\cdots(j_r)} = \begin{vmatrix} C^{(i_1)}_{(j_1)} & \cdots & C^{(i_1)}_{(j_r)} \\ \cdots & \cdots & \cdots \\ C^{(i_r)}_{(j_1)} & \cdots & C^{(i_r)}_{(j_r)} \end{vmatrix}. \tag{3}$$

As shown in [2; §2],

$$r_c = C^{(i)}_{(i)} . \tag{4}$$

Of considerable use is the fact that the rank tensor factors into the product of a contravariant numerical tensor and a covariant numerical tensor:

$$C^{(i_1)\cdots(i_r)}_{(j_1)\cdots(j_r)} = \frac{1}{c} \cdot C^{(i_1)\cdots(i_r)} \cdot C_{(j_1)\cdots(j_r)} \qquad (c \neq 0). \tag{5}$$

We should recognize that we may have a certain choice of freedom in regard to these factors. Let $C^{\beta_1 \beta_2 \cdots \beta_r}_{\gamma_1 \gamma_2 \cdots \gamma_r}$ be a non-zero component of the rank tensor; there will be more than one choice of the subscript labels and of the superscript labels. Once our choice has been made

$$c = C^{\beta_1 \cdots \beta_r}_{\gamma_1 \cdots \gamma_r} \neq 0; \quad C^{(i_1)\cdots(i_r)} = C^{(i_1)\cdots(i_r)}_{\gamma_1 \cdots \gamma_r}; \quad C_{(j_1)\cdots(j_r)} = C^{\beta_1 \cdots \beta_r}_{(j_1)\cdots(j_r)} . \tag{6}$$

Thus for $C^{(i)}_{(j)}$ the immanant [3] $I^{(i)}_{(j)} = \frac{1}{2}(\delta^{i_1}_{j_1}\delta^{i_2}_{j_2} + \delta^{i_1}_{j_2}\delta^{i_2}_{j_1})$, $r = 3$ for $n = 2$. Choose $(\beta_1, \beta_2, \beta_3) = (11, 12, 22)$ and $(\gamma_1, \gamma_2, \gamma_3) = (11, 12, 22)$. Then $c = \frac{1}{2}$, and the factorization of the rank tensor of $I^{(i)}_{(j)}$ is given by

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