RAMANUJAN'S FUNCTION $\tau(n)$

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The numerical function $\tau(n)$ defined by

$$(1) \qquad x\{(1-x)(1-x^2)\cdots\}^{24} = \sum_{n=1}^{\infty} \tau(n)x^n = x - 24x^2 + 252x^3 - \cdots$$

is generated by an infinite product which, for $x = \exp(2\pi i\omega_2/\omega_1)$, becomes one of the earliest known modular forms. It was not until 1916, however, that the discoveries of Ramanujan [9], [10; 136–162] called attention to the remarkable properties of the coefficients $\tau(n)$. A few of the conjectures and problems presented by Ramanujan are still unproved or unsolved in spite of considerable advances made in recent years in the theory of modular forms and their Fourier coefficients.

One of the most interesting facts about $\tau(n)$ noted by Ramanujan is that τ is a "multiplicative function"; that is,

$$(2) \qquad \tau(m)\tau(n) = \tau(mn),$$

for every coprime pair of integers (m, n). This fact permits the calculation of $\tau(n)$ from the values of $\tau(p\alpha)$, where p is a prime. These latter values, as noted by Ramanujan, form, with p fixed, a recurring series of the second order. In fact,

$$(3) \qquad \tau(p^{\alpha+1}) = \tau(p)\tau(p^{\alpha}) - p^{11}\tau(p^{\alpha-1}) \qquad (\alpha \geq 1).$$

Hence $\tau(p^{\alpha})$ may be found readily for $\alpha = 2, 3, \cdots$, once $\tau(p)$ is known. Properties (2) and (3) were first proved by Mordell [7] and have been generalized to other modular forms by Hecke [4]. As to $\tau(p)$, however, no explicit formula for it has yet been discovered. Ramanujan conjectured that

$$(4) \qquad |\,\tau(p)\,| < 2p^{11/2}$$

so that

$$p^{-11/2}\tau(p) = 2\cos\theta_p\,,$$

where θ_p is real.

This conjecture is still unproved and is called by Hardy [3; 169] the "Ramanujan hypothesis". Ramanujan regarded the truth of this hypothesis as "highly probable", although he verified only the first ten cases of primes $p < 30$. In seeking to disprove the Ramanujan hypothesis I have examined all primes $p < 300$, that is, the first 46 primes, as well as $p = 571$ and I find that in all

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