MINIMIZING AN INTEGRAL ON A CLASS OF CONTINUOUS CURVES

By George M. Ewing

Introduction. Hahn published in 1925 [2; 446–447] an example of a positive semi-definite positive quasi-regular variation problem which has no rectifiable minimizing curve. The problem admits a minimizing sequence (C_n) of rectifiable curves tending to a non-rectifiable curve C_0 and provided a suitable definition for the integral $J(C)$ is given for the case where C is non-rectifiable, then C_0 actually minimizes J on a class of continuous curves. Menger has obtained several existence theorems for problems in very general spaces which include Hahn's example. One of the chief difficulties is to obtain a convergent minimizing sequence. Menger uses the hypothesis that any sequence (C_n) on which his integral is bounded has a convergent sub-sequence [4; 1649], [5; 248], [6; 31] or that "lengths of comparison" are bounded on any class of curves on which values of the integral are bounded [7; 477].

We confine attention to variation problems in Euclidean spaces, using principally hypotheses of conventional type. We give two existence theorems (Theorems 2.1 and 3.1) which include the example of Hahn. A supplementary section is devoted to lower semi-continuity theorems.

1. **An abstract foundation.** Let A denote a given bounded closed subset of Euclidean n-space with points $p, q, r, \cdots$. In addition to the Euclidean metric denoted by pq we suppose given a second "distance" $\delta(p, q)$ which satisfies the axioms for a "metric space" [8; 467].

We assume that

(1) *for every* $\epsilon > 0$ *there exists* $e > 0$ *such that* $pq < e$ *implies* $\delta(p, q) < \epsilon$, e *not depending on* p *or* q.

The term *compact* will be used in the sense of self-compact. Such terms as compact, tend to, limit, etc. will always be understood to be based upon the metric pq unless the metric $\delta(p, q)$ is mentioned or it is quite clear from the context that $\delta(p, q)$ is intended.

The set A is compact in terms of pq and thus from (1) in terms of $\delta(p, q)$. Using the continuity of the metric [5; §4, (2)], we find that *for every* $\epsilon > 0$ *there exists* $e > 0$ *such that* $\delta(p, q) < e$ *implies* $pq < \epsilon$, e *not depending on* p *or* q.

Let C be a continuous curve on A with the continuous representation $x(t)$, $0 \leq t \leq 1$. The set of successive values $t_0 = 0, t_1, \cdots, t_{n-1}, t_n = 1$ furnishes a *partition* of C said to be of *norm* ν if $\max (t_i - t_{i-1}) = \nu$, $i = 1, \cdots, n$.

Denoting $x(t_i)$ by p_i $(i = 1, \cdots, n)$, we call the (possibly infinite) quantity

$$\text{l.u.b.} \sum_{i=1}^{n} \delta(p_{i-1}, p_i),$$

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