DIOPHANTINE EQUATIONS REDUCIBLE IN BIQUADRATIC FIELDS

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1. **Introduction.** In an earlier paper [2] the complete rational integer solution was obtained for certain Diophantine equations reducible in a single quadratic field. In this paper we deduce the complete solution in rational integers of some Diophantine equations by operating on multiplicative equations of the form $uN(X) = vN(Y)$ in certain biquadratic fields (for notation see §3). These equations are solved by an extension of the method used earlier [2], and it will be seen from these examples how this method can be used to solve completely Diophantine equations reducible in two or more quadratic fields. The idea is to operate in the field which has as its subfields those fields in which the equation is factorable. Thus, to solve the equation $x^3 + y^3 = u^2 + v^2$ we use the field $Ra(i, 3^{\frac{1}{2}})$. This field is an example of the so-called special Dirichlet biquadratic fields which we use in this paper.

2. **The special Dirichlet biquadratic fields** [1; 47–52]. These are the fields $Ra(i, m^{\frac{1}{2}})$ obtained by adjoining $i = (-1)^{\frac{1}{2}}$ and $m^{\frac{1}{2}}$ to the field of rational numbers, where m is a positive square-free rational integer different from ± 1. The numbers of $Ra(i, m^{\frac{1}{2}})$ are

$$X = a + ib + m^{\frac{1}{2}}c + im^{\frac{1}{2}}d,$$

where a, b, c, d are rational, and the integers [1; 25–26] of the field are of the form

$$X = \tfrac{1}{2}(r + is + m^{\frac{1}{2}}t + im^{\frac{1}{2}}u),$$

where $r \equiv u$, $s \equiv t \pmod 2$ if $m \equiv 3 \pmod 4$, $r \equiv t$, $s \equiv u \pmod 2$ if $m \equiv 1 \pmod 4$, r, s are even and $t \equiv u \pmod 2$ if $m \equiv 2 \pmod 4$.

The conjugates of X are the numbers X_1, X_2, X_3, respectively, obtained by changing in X the sign of i, the sign of $m^{\frac{1}{2}}$, the signs of both i and $m^{\frac{1}{2}}$; it follows then for each type of conjugate that the conjugate of a product is equal to the product of the conjugates of each factor. We also observe that the products XX_1, XX_2, XX_3 are numbers in the quadratic subfields $Ra(m^{\frac{1}{2}})$, $Ra(i)$, $Ra(im^{\frac{1}{2}})$, respectively. The norm $N(X)$ of a bi-quadratic number X is defined by $XX_1X_2X_3$, but if X is a quadratic integer, then $N_0(X)$ is the norm in the quadratic field (that is, $N_0(X) = XX_1$ or XX_2). If $N(X) = \pm 1$, then X is called a unit of the field.

3. **Notations.** Hereafter we shall adhere to the following notations. The letters $a, \cdots, g, s, t, \cdots, z$ will represent rational integers, while the remaining italic letters $h, j, \cdots, r$ (except m) will denote integers of $Ra(i)$. The capital letters $A, B, \cdots$ will represent integers of the field $Ra(i, m^{\frac{1}{2}})$; the Greek letters

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