THE BEHAVIOR OF EULER'S PRODUCT ON THE BOUNDARY OF CONVERGENCE

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1. **Introduction.** If $g(t)$, $0 \leq t < \infty$, is integrable (L) on every bounded interval $[0, T]$, let $M(g)$ denote the limit of the mean-value

$$M_T(g) = \frac{1}{T} \int_0^T g(t)\, dt \tag{1}$$

as $T \to \infty$, provided $M(g)$ exists as a *finite* limit.

It is known that $M(|f|^2)$ exists for $f(t) = 1/\zeta(1 + it)$, where $\zeta(s)$ is the Riemann zeta-function. The same has never been proved for the more fundamental function $f(t) = \arg \zeta(1 + it)$, that is, for the imaginary part of $\log \zeta(s)$ on the line $\sigma = 1$, where the phase $\arg \zeta(s)$ is understood to be defined for $\sigma \geq 1$ (but $s \neq 1$) by continuous variation of the initial phase $\arg \zeta(\infty + it) \equiv 0$. The appearance of additional difficulties introduced by the passage from $1/\zeta(1 + it)$ to $\arg \zeta(1 + it)$ is evident from the whole structure of the zeta-function. It will be shown in the present paper that the resulting complications can be avoided if the classical method is replaced by another approach. The principal difficulty appears to be a direct proof of $\limsup M_T(|f|^2) < \infty$ (in fact, not even $\limsup M_T(|f|) < \infty$ seems to have been established in the literature).

What will actually be proved is that $\log \zeta(1 + it)$ is almost periodic (B^2). Hence, the same is true of the imaginary part of $\log \zeta(1 + it)$. This implies, of course, much more than the mere existence of $M(|f|^2)$ for $f(t) = \arg \zeta(1 + it)$.

In order to delimit the nature of the difficulties involved, the case of the logarithmic derivative, $\zeta'/\zeta(1 + it)$, will first be treated. This case can be approached directly. On the other hand, the treatment of the case of $\log \zeta(1 + it)$ or of $\arg \zeta(1 + it)$ will substantially depend on a Fourier transformation of the problem.

The point is that the truth of Riemann's hypothesis is not assumed in either case. In fact, the results, along with their extensions from $\sigma = 1$ to $\sigma > \frac{1}{2}$, could readily be deduced from known consequences of Riemann's hypothesis.

Since such functions as $\zeta(1 + it)$ or $\zeta'/\zeta(1 + it)$ have non-integrable singularities at $t = 0$, the average, $M(g) \equiv \lim M_T(g)$, of such functions will have to be meant in the sense that the lower integration limit, $t = 0$, of (1) is replaced by some $t = a > 0$. This convention implies a corresponding proviso for the definition of almost periodicity (B^p). Needless to say, the resulting assertions are independent of the numerical value of a.

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