CESÀRO SUMMABILITY OF INDEPENDENT RANDOM VARIABLES

BY G. E. FORSYTHE

1. **Introduction.** The Cesàro methods for summability of sequences $\{x_k\}$ of real numbers have been studied extensively. It is known that the Cesàro means $\{C_r\}$ $(-1 < r < \infty)$ form a linear "scale": as $r \uparrow$, the means C_r become non-trivially stronger.

The objectives of the present paper are: (1) to introduce definitions for the Cesàro summability $\mathbf{C}_r$ $(r > 0)$ of sequences $\{\mathbf{x}_k\}$ of independent real-valued random variables, (i) to the limit 0, and (ii) to the normal distribution $\Phi(x)$, where

$$\Phi(x) = \frac{1}{(2\pi)^{\frac{1}{2}}} \int_{-\infty}^{x} e^{-\frac{1}{2}t^2}\, dt; \tag{1.1}$$

(2) to give necessary and sufficient arithmetical conditions in the two cases for a sequence $\{\mathbf{x}_k\}$ to be summable-$\mathbf{C}_r$; (3) to compare these conditions for varying r to see what becomes of the Cesàro "scale" in the two cases.

The first two objectives have been easily reached, since they amount only to extensions or specializations of general results of Feller [2] and [3] and Gnedenko [5], who employed characteristic functions in their proofs. The cases $r = 1$ reduce to the weak law of large numbers and the central limit theorem of probability, respectively.

The methods used to attain the third objective are arithmetical. In the case of summability to 0, objective (3) has been entirely attained for $r \geq 1$. For $0 < r < 1$ it has been attained except for an important unsolved problem.

In the matter of summability to the normal distribution, for the most part this paper deals only with the case of normal families $\{\mathbf{x}_k\}$ (defined in §3). For these the objective (3) has been achieved except for $\frac{1}{2} \leq r < 1$. Theorems 7.7 and 7.8 show for example that for $r \geq 1$, the summability methods $\mathbf{C}_r$ grow *non-trivially* stronger as $r \uparrow$.

Certain results on the more general Nörlund means over normal families will be reserved for a later paper.

In §§2, 3 and 4 are given the necessary definitions and notations. In §5 we consider the summability to 0 of symmetric random variables, and in §6 the general case is treated. In §7 summability to the normal distribution is treated.

2. **Summability definitions and special notations.** The symbol C will denote a generic constant independent of the parameters n or k of a given statement; but C may change value from one statement to the next. Thus $x_n \leq Cy_n$ $(n \geq 0)$ will imply that $5x_n \leq Cy_n$ $(n \geq 0)$.

Received January 25, 1943; presented to the American Mathematical Society May 3, 1941. The author was a Sigma Xi Fellow from Swarthmore College. This paper is part of a dissertation written at Brown University under Professors Tamarkin and Feller.