CERTAIN FUNCTIONS WITH SINGULARITIES ON THE UNIT CIRCLE

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1. Methods for determining the Fourier coefficients of modular forms of positive dimension have been given [3]. We shall be interested only in the simplest case where $F(\tau) = f(e^{2\pi i\tau})$ is of positive dimension and $f(x)$ is regular inside the unit circle except for a pole at $x = 0$. The unit circle is then a natural boundary for $f(x)$. The fact that $F(\tau)$ is modular is used only at the beginning of the determination of the coefficients. The behavior of $f(x)$ on the Farey arcs is found from the transformation formula for $F(\tau)$ and this yields the coefficients by an integration over the Farey arcs. Thus the function $f(x)$ is determined by its behavior on the Farey arcs, which means, roughly, by its asymptotic behavior near the "rational points" on the unit circle, $x = e^{2\pi ih/k}$.

It is clear that if a function $f(x)$, whether modular or not, is given with suitable behavior on the Farey arcs, then its Fourier coefficients can be determined in the same way. However, if we start merely with an arbitrary choice of the behavior on the Farey arcs, then the method can still be applied; we can find the coefficients a_m, but we are not able to assert that

$$f(x) = \sum_{m=-\mu}^{\infty} a_m x^m.$$

There may be no function, regular inside the unit circle except for a pole at $x = 0$, having the desired behavior and $\sum_{m=-\mu}^{\infty} a_m x^m$ may not have the required properties.

In this paper we shall consider this problem of whether $\sum_{m=-\mu}^{\infty} a_m x^m$ does have the desired behavior on the Farey arcs for certain kinds of assigned behavior. In §2 a particular example is considered and used to show that there are certain limitations on the degree with which we can expect $\sum a_m x^m$ to exhibit the required behavior. The considerations of that section show that we may just as well restrict our attention to functions without singularities within the unit circle.

Our main results can be summarized in the following theorems:

THEOREM 1. *Let r and γ be integers, $r \geq 0$, $\gamma > 0$, $\alpha \geq 0$, $\beta \geq 0$, $B \geq 0$, and let $d_{h,k}$ be a set of real or complex numbers defined for $0 \leq h < k$, $(h, k) = 1$, such that $d_{h,k} = O(k^{r\beta/\gamma-2-\epsilon})$, $\epsilon > 0$. Let*

$$f(x) = \sum_{m=0}^{\infty} a_m x^m$$

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