THE MAP-COLORING OF UNORIENTABLE SURFACES

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1. **Introduction.** By a *map* we understand a partition of an unbounded surface into F simply-connected regions ("faces") by means of E simple arcs ("edges") joining pairs of V points ("vertices"). It is well known [13; 149] that all maps on a given surface have the same *characteristic* $K = V - E + F$, and that the topological properties of an unbounded surface are completely determined by the value of K, except that when K is even the surface may be either orientable or unorientable.

A map can be specified by a combinatorial scheme which names the vertices of each face in proper cyclic order. If we can permute the names without altering the scheme as a whole, we say that the map is symmetrical: such permutations constitute its *symmetry group.*

A map is said to be colored if colors are assigned to its faces in such a way that any two faces with a common edge have different colors. There is evidently a number O_K (for each even $K \leq 2$) such that every map on an orientable surface of characteristic K can be colored with O_K colors, while at least one map cannot be colored with fewer; in other words, O_K colors are both sufficient and necessary. Thus the famous four color problem asks whether $O_2 = 4$ or 5. Similarly, there is a number U_K (for every integer $K \leq 1$) such that U_K colors are sufficient and necessary for an unorientable surface of characteristic K.

Heawood [10; 334], [1; 236] proved that every map on an orientable surface of characteristic $K < 2$ can be colored with $[F_K]$ colors, where

$$F_K = \tfrac{1}{2}(7 + (49 - 24K)^{\frac{1}{2}}). \tag{1}$$

As the same argument applies to an unorientable surface, this shows that

$$O_K \leq F_K, \qquad U_K \leq F_K \qquad\qquad (K < 2).$$

He conjectured that $O_K = [F_K]$ in every case. Heffter [11; 492–494] has verified this for $K = 0, -2, -4, -6, -8, -10$. But it is not always true that $U_K = [F_K]$; for Franklin [8; 368] has shown that $U_0 = 6$, although $F_0 = 7$. On the other hand, this failure may be exceptional, as the equation has been verified by Tietze [17; 158] for $K = 1$, by Kagno [12] for $K = -1, -2, -4$, and by Bose [2; 380] for $K = -5$. This paper fills a gap by establishing it also for $K = -3$. Moreover, the case $K = -5$ is discussed more fully, with the aid of models in Euclidean space of three or four dimensions.

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