THE ABSOLUTE SUMMABILITY OF POWER SERIES AND FOURIER SERIES

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1. **Introduction.** Let

$$f(z) = \sum_{n=0}^{\infty} c_n z^n \qquad (z = re^{i\theta}) \tag{1.0}$$

be a power series regular for $r < 1$. Various authors [2], [3], [4] have investigated the problems concerning the absolute convergence and the absolute summability of the series $\sum c_n e^{ni\theta}$, when $f(z)$ possesses the property

$$M_p(r, f') = \left(\frac{1}{2\pi}\int_0^{2\pi} | f'(re^{i\theta}) |^p \, d\theta\right)^{1/p} = O((1 - r)^{-1+k}) \tag{1.1}$$

as $r \to 1 - 0$, where $p \geq 1$, $0 < k < 1$. The purpose of the present paper is to study the same problems on the function $f(z)$ which satisfies, instead of (1.1), the condition

$$M_p(r, f') = O\left((1 - r)^{-1+1/p}\left(\log \frac{1}{1 - r}\right)^{-q}\right) \qquad (q > 1). \tag{1.2}$$

We prove that, if $1 < p \leq 2$ and $f(z)$ satisfies (1.2), then the series $\sum c_n e^{ni\theta}$ is absolutely convergent (Theorem 1). This is an extension of a theorem in [3] (Theorem 2 below). That the theorem ceases to be true for the case $p > 2$ is shown by the function

$$f(z) = \sum_{n=1}^{\infty} n^{-1} e^{ain \log n} z^n \qquad (a > 0),$$

which satisfies (1.2) for $p > 2$, but the series $\sum n^{-1}$ is divergent. We can however prove in this case that $\sum c_n e^{ni\theta}$ is absolutely summable (C, α) for every $\alpha > \frac{1}{2} - 1/p$ (Theorem 4).

If $u(\theta)$ is an integrable function, periodic with period 2π, let its Fourier series be

$$\tfrac{1}{2}a_0 + \sum_{n=1}^{\infty} (a_n \cos n\theta + b_n \sin n\theta).$$

Set $c_0 = \frac{1}{2}a_0$, $c_n = a_n - ib_n$ $(n > 0)$ in (1.0); then $f(z)$ is regular for $r < 1$. It can be proved that if $u(\theta)$ satisfies

$$\left(\int_0^{2\pi} | u(\theta + h) - u(\theta) |^p \, d\theta\right)^{1/p} = O\left(h^{1/p}\left(\log \frac{1}{h}\right)^{-q}\right) \qquad (p > 1,\ q > 1) \tag{1.3}$$

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