CHARACTERIZATION THEOREMS FOR INTEGRAL MEANS

By R. G. Helsel and P. M. Young

I. Summary

Because of the extensive applications of integral means [1], [3], [4], [5], [8] in problems connected with the area of surfaces, potential theory, and the calculus of variations, it is important to know many of their properties. The purpose of this paper is to present characterization theorems for the following types of integral means.

Definition. If $f(x)$ is summable on $0 \leq x \leq 1$ and $0 < h < \frac{1}{2}$ is fixed, then

$$f_h(x) = \frac{1}{2h} \int_{-h}^{h} f(x + \alpha)\, d\alpha = \frac{1}{2h} \int_{x-h}^{x+h} f(\xi)\, d\xi,$$

defined on $h \leq x \leq 1 - h$, is called the *integral mean of* $f(x)$.

Definition. If $f(x, y)$ is summable on the unit square $S_0 : 0 \leq x \leq 1$, $0 \leq y \leq 1$, and if $0 < h, k < \frac{1}{2}$ are fixed, then

$$f_h^k(x, y) = \frac{1}{4hk} \int_{-h}^{h} \int_{-k}^{k} f(x + \alpha, y + \beta)\, d\alpha\, d\beta = \frac{1}{4hk} \int_{x-h}^{x+h} \int_{y-k}^{y+k} f(\xi, \eta)\, d\xi\, d\eta,$$

defined on the rectangle $R_{hk} : h \leq x \leq 1 - h,\ k \leq y \leq 1 - k$, is called the *h-k-integral mean of* $f(x, y)$.

Definition. If $f(x, y)$ is summable on S_0 and if $0 < h < \frac{1}{2}$ is fixed, then

$$f_h(x, y) = \frac{1}{2h} \int_{-h}^{h} f(x + \alpha, y)\, d\alpha = \frac{1}{2h} \int_{x-h}^{x+h} f(\xi, y)\, d\xi,$$

defined for almost every y on the rectangle $R_{h0} : h \leq x \leq 1 - h,\ 0 \leq y \leq 1$, is called the *h-integral mean of* $f(x, y)$. Similarly,

$$f^k(x, y) = \frac{1}{2k} \int_{-k}^{k} f(x, y + \beta)\, d\beta = \frac{1}{2k} \int_{y-k}^{y+k} f(x, \eta)\, d\eta$$

is called the *k-integral mean of* $f(x, y)$.

Throughout the paper we work only with functions which belong either to class C^n or to class L^p.

Received September 11, 1942. The present paper contains a summary of related parts of two Ohio State University dissertations completed under the direction of T. Radó, namely, P. M. Young, *On the approximation of functions by integral means*, 1941 (unpublished), and R. G. Helsel, *A geometrical application of integral means*, 1942 (unpublished).