CONTRIBUTIONS TO THE PROBLEM OF GEÖCZE

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Introduction

0.1. Let S be a continuous surface defined by

$$S: \qquad z = f(x, y),$$

where $f(x, y)$ is defined and continuous on the closed unit square

$$I: \qquad 0 \leq x \leq 1, \qquad 0 \leq y \leq 1.$$

For a surface of this form the Lebesgue area is given by

$$A[S] = \inf \liminf A[P_n],$$

where $P_n : z = p_n(x, y)$, $x, y \in I$; $n = 1, 2, 3, \cdots$, is a sequence of polyhedra (*not necessarily inscribed*) such that $p_n(x, y) \to f(x, y)$ uniformly on I and the infimum is taken with respect to all such sequences of polyhedra. $A[P_n]$ denotes, of course, the area in the elementary sense of the polyhedra P_n.

0.2. Note that the above definition implies the existence of *some* sequence of polyhedra $P_n : z = p_n(x, y)$, $(x, y) \in I$, such that $p_n(x, y) \to f(x, y)$ uniformly in I and $A[P_n] \to A[S]$. There arises the question whether there exists a sequence of *inscribed* polyhedra with these properties. For the general case when $f(x, y)$ is merely continuous the answer to this question is as yet unknown.

Let $A^*[S]$ designate the quantity obtained by requiring that the polyhedra used in the definition of $A[S]$ be inscribed in S. Our question is then easily seen to be equivalent to *the problem of Geöcze* which consists of deciding whether or not $A^*[S] = A[S]$.

The work done so far on this problem (for previous results the reader is referred to [2] and [7]) is both extremely involved and inconclusive. The purpose of the present paper is to discuss the case when the function $f(x, y)$ is absolutely continuous in the Tonelli sense (cf. §1.3). Since absolute continuity of $f(x, y)$ in the sense of Tonelli is the necessary and sufficient condition that the area of the surface be given by the usual integral formula, this case is of considerable importance and it seems desirable to have simple and direct methods for it. This paper contains a presentation of two such methods, one based on approximation by integral means, and the other on recent results concerning convergence in area (cf. §1.9).

Both methods are entirely different from those used so far in the problem of

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