# FOURIER INTEGRALS

By Richard Bellman

The object of this paper is to extend a familiar gap theorem of Fourier series to Fourier integrals.

Let $f(x)$ be a function belonging to $L^p(0, \infty)$, $1 < p \leq 2$, and $(n_k)$ be a sequence satisfying the ordinary gap condition $n_{k+1}/n_k \geq \lambda > 1$.

Let

$$s_n(t) = \int_0^n f(x)e^{ixt}\, dx, \qquad F(t) = \underset{p'}{\text{l.i.m.}}\ s_n(t).$$

Then the theorem to be proved is

$$\lim_{k\to\infty} s_{n_k}(t) = F(t)$$

*almost everywhere.*

*Proof.* Let

$$\sigma_n(t) = \int_0^n \left(1 - \frac{x}{n}\right) f(x)e^{ixt}\, dx.$$

Then [2; 288]

$$\lim_{n\to\infty} \sigma_n(t) = F(t)$$

almost everywhere.

$$s_{n_k}(t) - \sigma_{n_k}(t) = \int_0^{n_k} \frac{xf(x)}{n_k} e^{ixt}\, dx.$$

By the analogue of the Young-Hausdorff theorem for Fourier series [1; 105],

$$\int_0^\infty |\, s_{n_k}(t) - \sigma_{n_k}(t)\, |^{p'}\, dt \leq M\left[\int_0^{n_k} \frac{|\, f(x)\, |^p x^p}{n_k^p}\, dx\right]^{p'/p}. \tag{1}$$

To show $s_{n_k}(t) - \sigma_{n_k}(t) = o(1)$, it is sufficient to show the convergence of

$$\sum_k |\, s_{n_k}(t) - \sigma_{n_k}(t)\, |^{p'}$$

almost everywhere or

$$\sum_k \int_0^\infty |\, s_{n_k}(t) - \sigma_{n_k}(t)\, |^{p'}\, dt.$$

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