THE SPAN OF MULTIPLY CONNECTED DOMAINS

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1. Let D_n be a domain in the z-plane, containing the point $z = \infty$ and bounded by n continua C_ν ($\nu = 1, \cdots, n$). A function $F(z)$ belongs to the family $\phi(D_n)$, if it is univalent and regular in D_n, the point $z = \infty$ excepted, at which it has the development

$$(1) \qquad F(z) = z + \frac{A_2}{z} + \frac{A_3}{z^2} + \cdots .$$

There exists a function $f(z) \subset \phi(D_n)$, mapping D_n on a domain bounded by slits parallel to the real axis. It can be characterized by the following extremal property [5]:

Among all functions $F(z) \subset \phi(D_n)$, $f(z)$ yields the maximal value of $\Re\{A_2\}$.

For the function $g(z) \subset \phi(D_n)$, mapping D_n on a domain bounded by slits parallel to the imaginary axis, an analogous result holds, namely:

Among all functions $F(z) \subset \phi(D_n)$, $g(z)$ yields the minimal value of $\Re\{A_2\}$. The functions

$$(2) \qquad f(z) = z + \frac{a_2}{z} + \frac{a_3}{z^2} + \cdots, \qquad g(z) = z + \frac{b_2}{z} + \frac{b_3}{z^2} + \cdots$$

will be called henceforth the *slit functions* of D_n.

The number

$$(3) \qquad S(D_n) = \Re\{a_2 - b_2\}$$

gives the breadth of the interval in which $\Re\{A_2\}$ varies for all functions of $\phi(D_n)$. $S(D_n)$ is a functional of D_n and will be called *the span of* D_n. The aim of this paper is to discuss some of its properties and to connect this number with other characteristics of the domain.

In this chapter some elementary theorems on $S(D_n)$ will be recalled. Let us remark first:

I. *The span is a non-increasing function of the domain.*

This theorem is obvious; for suppose $D_n \subset D'_m$, then $\phi(D'_m) \subset \phi(D_n)$; hence, the interval of variation of $\Re\{A_2\}$ is not smaller for D_n than for D'_m. This proves the assertion.

II. *The span of all domains which can be mapped on each other with the aid of univalent and normalized functions* $p(z) = z + p_1 + p_2/z + \cdots$ *is the same.*

Indeed, take $z = z(\zeta) = \zeta + \pi_1 + \pi_2/\zeta + \cdots$ as the univalent function mapping Δ_n in the ζ-plane on D_n in the z-plane. $F(z) \subset \phi(D_n)$ only if $F[z(\zeta)] -$

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