THE GENERAL TYPE OF SINGULARITY OF A SET OF $2n - 1$ SMOOTH FUNCTIONS OF n VARIABLES

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1. **Introduction.** Let a region R of n-space E^n, or more generally, of a differentiable n-manifold, be mapped differentiably into m-space E^m. If $m \geq 2n$, it is always possible [1; 818], [3], by a slight alteration of the mapping function f (letting also any finite number of derivatives change arbitrarily slightly), to obtain a mapping f^* which is everywhere regular. That is, for any p in R, and any set of independent vectors $u_1, \cdots, u_n$ in R at p, f^* carries these vectors into independent vectors. Here, vector equals the vector in "tangent space" equals the differential. As a consequence, some neighborhood U of p is mapped by f in a one-one way. The object of this paper is to determine what can be obtained by slight alterations of f in case $m = 2n - 1$. It turns out that any singularities may be made into a fixed kind. (It will be shown in other papers that any smooth n-manifold may be imbedded in $(2n)$-space, and may be immersed (self-intersections allowed) in $(2n - 1)$-space.)

There are two main theorems in the paper, roughly:

(a) We may alter f arbitrarily slightly, forming f^*, for which the singular points (points where f^* is not regular) are isolated, and such that a certain condition (C) below holds at each singular point. (The self-intersection may also be made simple; cf. [3; 655, (D)].)

(b) Let f^* satisfy the condition mentioned. Then for any singular point p, we may choose coördinate systems $x_1, \cdots, x_n$ in a neighborhood of p and $y_1, \cdots, y_{2n-1}$ in a neighborhood of $f(p)$ such that f^* is given exactly by the equations (4.2). Here, f^* must have many derivatives.

Remark. As a consequence, there is a slight deformation of E^{2n-1} which carries $f(U)$ (U a neighborhood of p) into the set of points given by (4.2).

The transformations in (b) may lower the class of f^* considerably; but if f^* is of class C^∞, or analytic, the transformations will be also. The condition mentioned in (a) is the following:

(C) There is a direction through p with the following properties: (C_1) f^* maps any vector in this direction into the null vector in E^{2n-1}, but maps any other vector at p into a non-null vector. (C_2) If $g(p')$ is the derivative of $f^*(p')$ in the direction given above, for p' near p, then there is no vector in E^{2n-1} which is the image both of a vector under f^* and a vector $\neq 0$ under g, both at p.

We may phrase the second condition as follows:

(C_2') Suppose a coördinate system is chosen in which the given vector is in the x_1-direction. Then

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