EXPONENTIAL CURVES

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I. Introduction

In order to illustrate by means of a specific example some of the more common occurrences in the theory of meromorphic [4] and analytic [5] curves, begun by the author in collaboration with Professor H. Weyl and successfully continued by L. Ahlfors [1], it seems worth while to undertake the study of certain special curves. These curves are on the one hand the normal exponential curves, defined in the complex k-dimensional space $\mathfrak{R} : \{x_1/x_0, \cdots, x_k/x_0\}$ by setting up

$$x_i = x_i(z) = \exp(\lambda_i z) \qquad (i = 0, 1, \cdots, k),$$

and on the other hand the general exponential curves, defined in an n-dimensional subspace $\mathfrak{R}' : \{w_1/w_0, \cdots, w_n/w_0\}$ of $\mathfrak{R}$ by setting up

$$w_i = w_i(z) = \sum_0^k a_{i\rho} \exp(\lambda_\rho z) \qquad (i = 0, 1, \cdots, n \leq k).$$

The arguments of geometric character which have been used by Pólya [2] and Schwengeler [3] for the investigation of finite exponential sums

$$F(z) = \sum_0^k \alpha_\rho \exp(\lambda_\rho z)$$

will allow us to sharpen the results gained from immediate application of the general theory of meromorphic curves. Some of the theorems found by these authors will be obtained again, imbedded, however, in a totally different but no less natural context. Our notation will correspond in all parts to the one employed in the papers [4] and [5].

1. **Preliminary remarks.** A normal exponential curve $\mathfrak{C}$ in the complex k-dimensional space $\mathfrak{R}$ is defined by setting up

$$x_i = x_i(z) = \exp(\lambda_i z) \qquad (i = 0, 1, \cdots, k),$$

where the λ_i are arbitrary complex numbers. Denoting its Wronskian determinant by

$$W = [x(z) \cdots x^{(k)}(z)]_{01\cdots k} = [1\lambda \cdots \lambda^k]_{01\cdots k} \exp(z \sum_0^k \lambda_\rho),$$

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