RIEMANN'S HYPOTHESIS AND HARMONIC ANALYSIS

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1. The results of this paper center around a problem which in a certain sense is the inverse of a question treated previously [7], [9]. This previous question concerns the harmonic analysis of the remainder term of the prime number theorem $\psi(x) \sim x$ if Riemann's hypothesis $\psi(x) = x + O(x^{\frac{1}{2}+\epsilon})$ is assumed. The result was that the reduced remainder term, $\{\psi(x) - x\}/x^{\frac{1}{2}}$, when considered as a function of $u = \log x$, turns out to be almost periodic (B^2) and that the sequence of the frequencies λ_n of the Fourier series $\sum a_n e^{i\lambda_n u}$ is identical with the sequence of the imaginary parts of the non-trivial zeros $s = \frac{1}{2} + i\lambda_n$ of $\zeta(s)$, while the amplitudes a_n are given by $a_n = -(\frac{1}{2} + i\lambda_n)^{-1}$. Thus the reduced remainder function of the prime number theorem, a function defined in terms of the logarithms of all the prime powers, leads to a Fourier analysis depending on the non-trivial zeros of the ζ-function (needless to say, such an almost periodic behavior is impossible unless Riemann's hypothesis is true).

In the sequel, the existence of the corresponding Fourier analysis of the remainder term of the classical asymptotic formula for $N(T)$ will be deduced, where $N(T)$ is, under Riemann's hypothesis, the number of zeros on the segment $0 < t \leq T$ of the line $\sigma = \frac{1}{2}$. The sequence of the Fourier frequencies of this remainder function, a function defined in terms of the non-trivial zeros of $\zeta(s)$, turns out to be identical with the sequence of logarithms of all the prime powers, which determine the Fourier amplitudes also. Thus the resulting harmonic analysis can be thought of as an inversion of the one obtained previously for the remainder term of the prime number theorem.

The Fourier character of the reciprocal mates supplies a sharp formulation of the mysterious connection between the non-trivial zeros of $\zeta(s)$ and the logarithms of all the prime powers. However, the resulting precise formulation cannot be expected to contribute much to the solution of the mystery.

2. Let $f(t)$ be a function defined for $0 \leq t < \infty$ and integrable (L) on every bounded interval, $0 \leq t \leq T$. If the average

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T f(t)\, dt$$

exists as a *finite* limit, it is denoted by $M(f)$. If $M(f_\lambda)$, where $f_\lambda(t) = f(t)e^{-i\lambda t}$, exists not only for $\lambda = 0$ but for every real number λ, the function $f(t)$ will be said to be of class (F). I do not know whether a function $f(t)$ can or cannot be so weird that $M(f_\lambda)$ exists and does not vanish on a non-enumerable λ-set. Let, therefore, $f(t)$ be said to possess a Fourier expansion if it is of class (F) and

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