A SPECIAL TYPE OF CONFORMAL MAP

By J. W. Green

1. Let G_z be the unit circle $|z| < 1$ in the z-plane and E a set of points on the circumference C_z of G_z which is open with respect to C_z. The set E is the sum of a finite or infinite number of open intervals and can be written $\sum \alpha_i$, where α_i is an open interval, the end points of which are not points of E. The summation runs from 1 to N or from 1 to ∞ according as the number of α_i is finite or infinite. F is the complement of E with respect to C_z and is assumed to be not void. It is desired to find a conformal mapping $w = w(z)$ of G_z on the unit circle G_w with certain radial slits removed, such that points of F map continuously on points of $|w| = 1$, and the intervals α_i map continuously on the radial slits. In this paper a radial slit means a line segment in the circle along a radius, of length less than the radius, and terminating at the circumference.

In terms of harmonic functions, it is desired to find a harmonic function $u(z)$ in G_z, which becomes infinite like $-\log r$ at 0, such that u vanishes continuously on F and $\partial u/\partial r$ vanishes continuously on E; i.e., a Green's function for the mixed boundary value problem with respect to F and E. Such a $u(z)$ is given by $-\log |w(z)|$. This Green's function can be obtained from the ordinary Green's function in the following way: let $g(0, P)$ be the Green's function with pole at 0 for the region exterior to F, and $g(\infty, P)$ the one with pole at infinity. Set $u(P) = g(0, P) + g(\infty, P)$. At regular points of F, $u(P) = 0$. On E, $\partial u/\partial r = 0$, since $u(P) = u(P^*)$, where P and P^* are inverse in C_z. Thus, if all the points of F are regular, $u(P)$ is the desired harmonic function in G_z. If F has irregular points, it is impossible to get a Green's function vanishing continuously on F. In the present paper, the described mapping function and thus the corresponding Green's function will be obtained under restrictive conditions on E and F as the solution of an extremal problem in the theory of analytic functions.

The set F can be written in the form $F = (C_z - \overline{E}) + (\overline{E} - E)$, where $(C_z - \overline{E})$ is open or null. This latter case will henceforth be prohibited, i.e., F will be assumed to have some interior points. Thus, we can write $(C_z - \overline{E}) = \sum \beta_i = \beta$, where β_i is an open interval. The set $(\overline{E} - E)$ is nowhere dense. Consider the class Γ of functions $f(z)$ with the following properties:

(1) $f(z)$ is analytic and schlicht in G_z;

(2) $|f(z)| < 1$ in G_z and $f(0) = 0$;

(3) $f(z)$ is continuous on $G_z + \beta$ and $|f(z)| = 1$ on β.

The class Γ is not void since it contains $f(z) \equiv z$. The principal theorem of this paper is as follows:

Theorem 1. *Let the set β and the class Γ be as described above. There exists a function $g(z)$ contained in Γ such that $|g'(0)| \leq |f'(0)|$ for every f of Γ. The*

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