# ON A THEOREM OF ZYGMUND

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1. The following theorem is due to Zygmund:

*If a continuous function $f(x)$ of period $2\pi$ is of bounded variation and has a modulus of continuity $\omega(\delta)$ such that the series $\sum n^{-1} [\omega(n^{-1})]^{\frac{1}{2}}$ converges, then the Fourier series of $f(x)$ is absolutely convergent.*

See [2]. Although the result is not stated in this form, the above statement is an immediate consequence of the argument used.

It is an outstanding question to know if this result can be strengthened, that is to say, if the exponent $\frac{1}{2}$ of this theorem can be replaced by a greater one. That it can certainly not be replaced by $1 + \epsilon$ for $\epsilon > 0$ is almost trivial, as can be seen by the consideration of the series $\sum \sin nx(n \log n)^{-1}$. The purpose of this paper is to prove that the exponent $\frac{1}{2}$ in Zygmund's theorem is the best possible one, the theorem becoming false if we replace the exponent $\frac{1}{2}$ by $\frac{1}{2} + \epsilon$, $\epsilon$ being any positive number.

2. We shall make use, in our proof, of certain singular monotonic functions of the Cantor type (i.e., continuous non-decreasing functions which are constant in each interval contiguous to a perfect set of measure zero) which have been described in a previous paper [1]. For the sake of completeness we shall give a brief account of these functions here.

*Symmetrical perfect sets of order d.* Letting $d$ be an integer $\geq 1$ and $\xi$ a positive number less than $(d + 1)^{-1}$, we mark, on an interval $(A, B)$ of length $B - A = L$, $d + 1$ non-overlapping "white" intervals of length $L\xi$, such that the first interval has its left end point in $A$, and the last interval has its right end point in $B$, the $d + 1$ "white" intervals being separated by $d$ "black" intervals of equal length

$$L\frac{1 - (d + 1)\xi}{d}.$$

Such a dissection will be called a $(d, \xi)$ dissection of the given interval.

Let us make now a $(d, \xi_1)$ dissection of the interval $(0, 2\pi)$ and let us remove the black intervals. In a second step we make a $(d, \xi_2)$ dissection of each white interval left, and we remove the black intervals. Proceeding in the same way, after $p$ operations, we have $(d + 1)^p$ intervals left; each of them is of length $2\pi\xi_1\xi_2 \cdots \xi_p$ ; their left end points have abscissas given by the formula

$$(1) \qquad 2\pi\left[\theta_1 \frac{1 - \xi_1}{d} + \theta_2 \frac{\xi_1(1 - \xi_2)}{d} + \cdots + \theta_p \frac{\xi_1\xi_2 \cdots \xi_{p-1}(1 - \xi_p)}{d}\right],$$

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