SEQUENCES OF STIELTJES INTEGRALS, II

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1. In his investigation of singular integrals, Lebesgue [8] was led to consider the following problem: Given a sequence of functions $G_n(x)$ $(n = 1, 2, \cdots)$ and a family F of functions $f(x)$ such that the integrals

$$\int_a^b f(x)G_n(x)\,dx \qquad (n = 1, 2, \cdots) \tag{1.1}$$

exist, determine conditions on $G_n(x)$ which are necessary and sufficient for the convergence to zero of the sequence (1.1). Lebesgue found solutions to this problem for those families F which naturally present themselves in analysis, and he applied his results to the theory of expansions of functions, while later some of these results were utilized also in the theory of weak convergence of sequences of linear functionals in Banach spaces [3]. These applications as well as the form of (1.1) suggest an extension of the above problem resulting from the replacement of the sequence (1.1) by the sequence

$$\int_a^b f(x)\,dg_n(x) \equiv J_n(f) \qquad (n = 1, 2, \cdots), \tag{1.2}$$

where the Stieltjes integrals (1.2) are assumed to exist in some specified sense.

This extended problem has been fully treated in the case of continuous $f(x)$, while a partial treatment, involving only sufficiency conditions for the required convergence of (1.2), has been given in a few other cases [9]. (To the references given in [9] should be added [2], which was not known to the author at the time of writing the paper [9]. Theorem 3 of [2], completed with the statement given therein in footnote (8), though seemingly less general than Theorem 4 of [9], can be shown to be essentially equivalent to that theorem.) The work relating to the case of continuous $f(x)$ is discussed in detail in [4; Chapter 7]. The list of references given therein seems to be complete except for the omission of [6].

In applications to the theory of linear functionals in Banach spaces, the case which presents greatest interest is that of continuous $f(x)$, which explains in part why this is the only case in the stated problem which has so far been studied. For other applications, for instance that to the theory of expansions of functions in terms of a set of functions, orthonormal in the sense of integration with respect to a given function, a general treatment of the problem appears to be desirable. Such a treatment is the object of the present study, which will be presented in several parts. In these it is hoped to include also a few of the more immediate applications.

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