THE APPROXIMATION OF ARBITRARY BIUNIQUE TRANSFORMATIONS

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In this paper, we shall show that every biunique correspondence of a square into itself may be approximated, in a certain sense, by a correspondence which is biunique and bicontinuous, i.e., by a homeomorphism. This theorem may be regarded as a sequel to one given by Franklin and Wiener [3] on the approximation of homeomorphisms by analytic transformations. It bears a relationship to various theorems in the literature on arbitrary real functions [1], [2], [5] similar to that which the result of Franklin and Wiener bears to the theorem of Weierstrass on the approximation of continuous functions by polynomials.

Definition. A biunique correspondence κ of a closed square S into itself will, for a given positive ϵ, be said to be *ϵ-approximated* by another identically-conditioned correspondence κ^* if there exist two sets A and B in S, each of relative exterior measure greater than $1 - \epsilon$ with respect to S, such that for every point p of A the distance between p^1 and p_1 is less than ϵ, and for every point p of B the distance between p^{-1} and p_{-1} is less than ϵ, where p^1, p^{-1}, p_1 and p_{-1} are the respective mates of p according to κ, κ^{-1}, κ^*, and κ^{*-1}; and κ^{-1}, κ^{*-1} are the respective inverses of κ, κ^*.

We now prove the result we have in mind:

Theorem I. *Every biunique correspondence κ of a square S into itself may, for every $\epsilon > 0$, be ϵ-approximated by a homeomorphism.*

Proof. We assume, as we may, that S is the unit square. Subdivide S into n squares S_i, $i = 1, \cdots, n$, each of diagonal length less than ϵ. If a point is on the boundary of two or more of the S_i, we assign it, at random, to just one of these squares; as a consequence, every point of S belongs to just one of the S_i. The image of S_i, by κ, is a point set T_i, and by κ^{-1}, a point set T_i^{-1}. Since κ is a biunique correspondence, the n sets T_i constitute a disjoint subdivision of S. The set T_1, except for a subset E_1 of arbitrarily small exterior measure, say $< \epsilon/2n$, may be enclosed in the sum R_1 of a finite number $R_{11}, R_{12}, \cdots, R_{1\mu_1}$ of non-overlapping, non-abutting, closed rectangles, such that the relative exterior measure of T_1 in R_1 exceeds $1 - \frac{1}{2}\epsilon$. The set of points of T_2 not in R_1 may, except for a subset E_2 of exterior measure $< \epsilon/2n$, be enclosed in the sum R_2 of a finite number of closed rectangles $R_{21}, R_{22}, \cdots, R_{2\mu_2}$, which neither overlap nor abut one another nor any component of R_1, such that the relative exterior measure of T_2 in R_2 exceeds $1 - \frac{1}{2}\epsilon$. Continuing as indicated, we define R_3 to be a sum of closed rectangles $R_{31}, R_{32}, \cdots, R_{3\mu_3}$, which neither overlap nor abut one another nor any component rectangle of $R_1 + R_2$, R_3

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