SOME GEOMETRIC INEQUALITIES

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1. The problem treated in this note was originally formulated by R. Salem and D. C. Spencer in connection with a number-theoretical investigation. Consider a plane domain Γ contained in the unit circle; suppose that the intersection of Γ with any straight line has a measure not exceeding a fixed constant $\delta < 1$. What can be said about the measure M of Γ? If Γ is convex, its area is obviously not greater than $\delta^2\pi$. It is less trivial that, for a convex domain Γ,

$$M \leq \tfrac{1}{4}\delta^2\pi \tag{1}$$

and that the sign of equality holds only if Γ is the interior of a circle; this result was proved by different methods by Bieberbach [1] and Kubota [3]; see also [2], particularly §§44, 54.

It has been widely conjectured that in general $M = O(\delta^2)$ as $\delta \to 0$, or at least that $M = o(\delta)$. Now a simple application of Fubini's theorem shows immediately that necessarily

$$M < 2\delta. \tag{2}$$

It will be proved in the sequel that (2) is the *best* result. In fact, we shall construct a domain Γ (consisting of a finite number of annuli) such that its intersection with any straight line of the plane has a total length not exceeding δ, whereas for its area M we have

$$M > 2\delta(1 - \delta^2\pi^{-2} - \epsilon), \tag{3}$$

where $\epsilon > 0$ is arbitrarily small. Thus 2δ is the best asymptotic estimate for the maximum of the area. The $1/\pi^2$ which multiplies δ^2 is, of course, not the best possible. Our construction can easily be refined, but this seems to be of no interest.

In §4 the above mentioned theorem of Bieberbach and Kubota will be proved in a new simple way which will make the result appear almost trivial. Actually the new proof is even slightly more general.

The generalization of the last result to n dimensions is straightforward. In order to solve the problem of the best estimate in the general case and in n dimensions, we shall (§5) formulate, and solve, a more general and purely analytic problem; it will be seen that our problem actually reduces to an inequality between two integrals.

2. Let $0 < \delta < 1$ be given and denote by N an arbitrarily large but fixed integer. Put

$$N' = [N(1 - \delta^2\pi^{-2})^{\frac{1}{2}}]. \tag{4}$$

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