THE CALCULUS OF VARIATIONS IN ABSTRACT SPACES

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The classical non-parametric problem of the calculus of variations deals with arcs defined by functions $y_p(x)$ $(p = 1, 2, \cdots, n)$. In the present paper we allow the parameter p to range over a quite arbitrary set $\mathfrak{P}$, and seek conditions that an arc $y_p(x)$ render an integral of the form

$$I = \int_{x_1}^{x_2} f[x, y(x), y'(x)]\, dx$$

a minimum in a class of admissible arcs. It is shown that this more general problem has a theory as complete as that of the classical problem. In a subsequent paper the author will take up the problem of Bolza in this general environment. In the first six sections, analogues of the familiar four necessary conditions are obtained. In §7 the sufficiency proofs are made, and in §8 the relation beteen conjugate points and the positiveness of the second variation is discussed.

1. **Formulation of the problem.** We shall use the notation $\mathfrak{R}$ to represent the set of real numbers, $\mathfrak{P}$ an arbitrary class of elements p, and $\mathfrak{V}$, an arbitrary Banach space of functions v on $\mathfrak{P}$ to $\mathfrak{R}$. It will be supposed that $(\mathfrak{R}, \mathfrak{V}, \mathfrak{V})_0$ is a region of the composite space $(\mathfrak{R}, \mathfrak{V}, \mathfrak{V})$ of sets (r, v, w) and that f on $(\mathfrak{R}, \mathfrak{V}, \mathfrak{V})_0$ to $\mathfrak{R}$ is a function of class C^{iv} uniformly on $(\mathfrak{R}, \mathfrak{V}, \mathfrak{V})_0$. See [4], [5], [8]. An *admissible arc* $y(x)$ is a continuous function y on (x_1, x_2) to $\mathfrak{V}$ which consists of a finite number of pieces on each of which $y'(x) \equiv \delta_x y(x; 1)$ exists and is continuous [5; 164] and such that each set $(x, y(x), y'(x))$ is in the fundamental region $(\mathfrak{R}, \mathfrak{V}, \mathfrak{V})_0$. An *admissible variation* is a function on an interval (x_1, x_2) having the continuity and differentiability properties of an admissible arc.

Our problem may then be formulated as that of finding in the class of admissible arcs joining two fixed points (x_1, y_1) and (x_2, y_2) one which minimizes the integral

$$I(C) = \int_{x_1}^{x_2} f[x, y(x), y'(x)]\, dx. \tag{1.1}$$

To carry through our analysis we shall suppose that there exists a mapping $(\nu_p \mid p \;\varepsilon\; \mathfrak{P})$ of $\mathfrak{P}$ onto a bounded subset $\mathfrak{V}_1$ of $\mathfrak{V}$ such that the linear extension of $\mathfrak{V}_1$ is dense in $\mathfrak{V}$, the limits being taken in the Moore-Smith sense; i.e., to each v in $\mathfrak{V}$ there corresponds a set of real numbers $a_{p\tau}$, where τ is a finite subset of

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