EFFECTIVE PARAMETERS

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1. **Introduction.** Under the classical treatment of essential parameters [1], [3] for a set of functions $f_1(x_1, \cdots, x_n, \alpha_1, \cdots, \alpha_m), f_2(x, \alpha), \cdots, f_r(x, \alpha)$, the parameters $\alpha_1, \alpha_2, \cdots, \alpha_m$ are called *essential* if there do not exist $m - 1$ functions of them, say $A_1(\alpha), \cdots, A_{m-1}(\alpha)$, and r functions $F_1, \cdots, F_r$ such that

$$(1.1) \qquad f_i(x; \alpha_1, \cdots, \alpha_m) \underset{(x,\alpha)}{\equiv} F_i[x; A_1(\alpha), \cdots, A_{m-1}(\alpha)] \qquad (i = 1, \cdots, r).$$

The treatment includes an algorithm for determining the number of essential parameters in terms of which the given functions can be expressed. If that number is $m - 1$, for example, then the final set of functions are the F_i in (1.1).

A weakness in this treatment is that the functions F_i are in general unknown. (The fact that the A's are in general unknown functions of the α's seems of little consequence.) In this paper we show that the same algorithm leads to like results, where in place of identities of the form (1.1) we have similar identities in the x's only. But in our case *the F's are known functions*; in fact, $F_i = f_i$ with $\alpha_j = A_j$ for a known set of subscripts j, with the remaining α's constant (Theorem 8.1). All r-tuples $f_1, f_2, \cdots, f_r$ of functions of the x's are obtained in this way which can be obtained by varying all the α's, and, incidentally, without duplication.

In all the work certain "singular" points must be avoided, both in the classical and in the present treatment. A discussion of the singular points is included.

A second deficiency in the classical treatment is the failure to establish a minimum number of parameters in terms of which the given r-tuple of families of functions of $(x_1, \cdots, x_n)$ can be expressed by means of differentiable functions; for this is done only under the restriction that the new parameters be functions of the old. In this paper we show without restriction, except as to the class of the given functions and the singularity of the points, that no smaller number of parameters can be sufficient (Theorem 10.5).

2. **Preliminaries.** Let functions $f_1(x_1, \cdots, x_n, \alpha_1, \cdots, \alpha_m) \equiv f_1(x, \alpha)$, $f_2(x, \alpha), \cdots, f_r(x, \alpha)$ be given, real and of class C^{N+3}, in a neighborhood of $(x_1^0, \cdots, x_n^0, \alpha_1^0, \cdots, \alpha_m^0)$; N is to be specified later; $n, m, r \geq 1$. If the functions are analytic, no restriction to real functions is necessary. We consider the f's as functions of $(x_1, \cdots, x_n)$, with $\alpha_1, \cdots, \alpha_m$ as parameters. Let us denote by $\{f(x, \alpha)\}$ the r-tuple $f_1(x, \alpha), \cdots, f_r(x, \alpha)\}$ of functions. In the case of a set containing only one function, we omit the braces.

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