INTEGRAL FORMULAS IN CROFTON'S STYLE ON THE SPHERE AND SOME INEQUALITIES REFERRING TO SPHERICAL CURVES

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Introduction. Several integral formulas referring to convex plane curves, notable for their great generality, were obtained by W. Crofton in 1868 and successive years from the theory of geometrical probability [6], [7], [8], [9], [10].

A direct and rigorous exposition of Crofton's principal results, adding some new formulas, was made in 1912 by H. Lebesgue [12]. Another systematic exposition of Crofton's most interesting formulas, together with the generalization of many of them to space, is found in the two volumes on integral geometry by Blaschke [2].

The purpose of the present paper is to give a generalization of Crofton's formulas to the surface of the sphere. This is what we do in part I. We find further integral formulas on the sphere (for instance, (16), (17), (20), (21)) which have no equivalent in the plane. Other formulas, if we consider the plane as the limit of a sphere whose radius increases indefinitely, give integral formulas referring to plane convex curves (e. g., (34), (35)) which we think are new.

In part II, with simple methods of integral geometry [2], we obtain three inequalities referring to spherical curves. Inequality (38) is the generalization to the sphere of an inequality that Hornich [11] obtained for plane curves. (52) and (58) contain the classical isoperimetric inequality on the sphere. Finally, inequality (61) gives a superior limitation for the "isoperimetric deficit" of convex curves on the sphere.

I. FORMULAS IN THE CROFTON STYLE ON THE SPHERE

1. **Notation and useful formulas.** The element of area on the sphere of unit radius will be represented by $d\Omega$; that is, if θ and φ are the spherical coördinates of the point Ω, we have

$$d\Omega = \sin \theta \, d\theta \, d\varphi. \tag{1}$$

A great non-directed circle C on the same sphere of unit radius can be determined by one of its poles, that is, by either of the extremities of the diameter perpendicular to it. Since $d\Omega$ is the element of area of one of these extremities, the "density" for measuring sets of great circles on the sphere is [2; 61, 80]

$$dC = d\Omega; \tag{2}$$

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