THE POISSON INTEGRAL REPRESENTATION OF FUNCTIONS WHICH ARE POSITIVE AND HARMONIC IN A HALF-PLANE

By L. H. Loomis and D. V. Widder

A theorem of Herglotz states that a function is positive and harmonic in a circle if and only if it can be expressed as a Poisson-Stieltjes integral with non-decreasing integrator function (see, for example, [1; 571] for a proof and for a reference to the original paper). A corresponding result was stated and proved by S. Verblunsky [2] for a half-plane. The proof was obtained by transformation of the half-plane into a circle. We propose to give here an independent proof without use of any such transformation. We obtain incidentally an interesting uniqueness result, Theorem 3, which seems not to have been stated explicitly before. The proof assumes no more recondite knowledge of a harmonic function than that it cannot have a minimum inside its region of definition.

Theorem 1. *If* $\varphi(x)/(1 + x^2) \in L$ *in* $(-\infty, \infty)$ *and is continuous at* x_0 , *then*

$$F(x, y) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{y}{y^2 + (t - x)^2} \varphi(t)\, dt \tag{1}$$

is harmonic for $y > 0$ *and* $F(x_0 , 0 +) = \varphi(x_0)$.

As $|t| \to \infty$, the expression $(t^2 + 1)/(t - x)^2$ approaches 1, and so has a finite upper bound $M_\delta(x)$ over $|t - x| \geq \delta$. Hence for $y \neq 0$

$$\int_{x+\delta}^{\infty} \frac{y\varphi(t)}{y^2 + (t - x)^2}\, dt \ll yM_\delta(x) \int_{x+\delta}^{\infty} \frac{|\varphi(t)|}{1 + t^2}\, dt.$$

It is thus clear that the integral (1) converges for $y > 0$. It represents a harmonic function there since the integrand is harmonic for each fixed t. Since

$$\frac{1}{\pi} \int_{-\infty}^{\infty} \frac{y}{y^2 + (x - t)^2}\, dt = 1,$$

we also have, for any $\delta > 0$,

$$|F(x_0 , y) - \varphi(x_0)| \leq M_\delta(x_0) \frac{y}{\pi} \left(\int_{-\infty}^{x_0-\delta} + \int_{x_0+\delta}^{\infty} \right) \frac{|\varphi(t) - \varphi(x_0)|}{1 + t^2}\, dt + \underset{|t-x_0|\leq\delta}{\text{u.b.}} |\varphi(t) - \varphi(x_0)|.$$

Hence

$$\overline{\lim_{y\to 0+}} |F(x_0 , y) - \varphi(x_0)| \leq \underset{|t-x_0|\leq\delta}{\text{u.b.}} |\varphi(t) - \varphi(x_0)|.$$

Since the right side approaches zero with δ, the theorem is proved.

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