THEORY OF EQUIVALENCE RELATIONS

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The present paper contains an extensive analysis of the theory of equivalence relations. A large number of new results have been obtained and it seems of particular interest that several of these connect rather diverse mathematical fields. But this theory appears to be of importance also for various other reasons. It may be mentioned, for instance, that a greater part of the theory consists in a study of the structure of all equivalence relations over a set. This structure does not satisfy the Dedekind condition and it is of a sufficiently general type to afford a useful example for arbitrary structure theory. Several problems solved for the structure of equivalence relations give indications of suitable methods of attack for analogous problems in the general case. As another justification for the theory of equivalence relations one may mention the fact that it yields an example for the much more general theory of mathematical relations which I have been studying for some time.

The main contents of the paper are as follows. In the first chapter one finds the simplest properties of *equivalence relations*, their representation as *partitions* or *fields of sets* over the basic set S, and it is shown that they form a complete structure. All these results were obtained by Garrett Birkhoff for the case of a finite set S, but the extension to arbitrary sets entails no particular difficulties. D. König made the important observation that there is associated a special so-called *pair graph* with the union of any two equivalence relations. Next the *Dedekind relation* and the *distributive law* are studied in the structure of equivalence relations and it is of interest that the results can be interpreted as properties of the union graph of two relations; for instance, the Dedekind relation is satisfied in one form if and only if the graph is a *tree*. The law of isomorphism is discussed by means of the Dedekind law. This leads to problems investigated by the Dubreils on the theorem of Jordan-Hölder and isomorphic chain refinements.

The structure of equivalence relations is *complemented* or even *completely complemented*. Furthermore, it possesses a special type of complements which I have called *Dedekind complements*. The construction of a Dedekind complement corresponds to a choice of representatives in the sets of the partition. The well-known problem of finding *common representatives* for two partitions may therefore be formulated structurally as a determination of common Dedekind complements. A few illustrations of the application of the theory to groups are given. It is shown that the existence of common representatives for right and left co-set expansions depends on the fact that the right and left co-set expansions always define so-called *commuting equivalence relations*.

In the third chapter some connections between partitions and *correspondences*

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